On the right-hand side we have

\[
(\theta + \alpha)(zF'(z) + bF(z)) = z \frac{d}{dz} (zF'(z) + bF(z)) + a(zF'(z) + bF(z))
\]

\[
= zF'(z) + z^2 F''(z) + bzF'(z) + azF'(z) + abF(z).
\]

Equating the two sides tells us that

\[
z(1-z)F''(z) + (c - z(a + b + 1))F'(z) - abF(z) = 0. \quad (5.108)
\]

This equation is equivalent to the factored form (5.107).

Conversely, we can go back from the differential equation to the power series. Let's assume that \( F(z) = \sum_{k \geq 0} t_k z^k \) is a power series satisfying (5.107). A straightforward calculation shows that we must have

\[
\frac{t_{k+1}}{t_k} = \frac{(k+a_1) \cdots (k+a_m)}{(k+b_1) \cdots (k+b_n)(k+1)},
\]

hence \( F(z) \) must be \( t_0 F(a_1, \ldots, a_m; b_1, \ldots, b_n; z) \). We've proved that the hypergeometric series (5.76) is the only formal power series that satisfies the differential equation (5.107) and has the constant term 1.

It would be nice if hypergeometrics solved all the world's differential equations, but they don't quite. The right-hand side of (5.107) always expands into a sum of terms of the form \( \alpha_k z^k F^{(k)}(z) \), where \( F^{(k)}(z) \) is the kth derivative \( D^k F(k) \); the left-hand side always expands into a sum of terms of the form \( \beta_k z^k \) with \( k > 0 \). So the differential equation (5.107) always takes the special form

\[
z^{n-1}(\beta_n - z\alpha_n)F^{(n)}(z) + \cdots + (\beta_1 - z\alpha_1)F'(z) - \alpha_0 F(z) = 0.
\]

Equation (5.108) illustrates this in the case \( n = 2 \). Conversely, we will prove in exercise 6.13 that any differential equation of this form can be factored in terms of the 4 operator, to give an equation like (5.107). So these are the differential equations whose solutions are power series with rational term ratios.

Multiplying both sides of (5.107) by \( z \) dispenses with the D operator and gives us an instructive all-4 form,

\[
\theta(\theta + b_1 \cdots (4 + b_n) F = z(\theta + a_1) \cdots (\theta + a_m) F. \quad (5.109)
\]

The first factor 4 \( = (\theta + 1 - 1) \) on the left corresponds to the \( (k+1) \) in the term ratio (5.81), which corresponds to the \( k! \) in the denominator of the kth term in a general hypergeometric series. The other factors \( (4 + b_i - 1) \) correspond to the denominator factor \( (k+b_i) \), which corresponds to \( b_i^k \) in (5.76). On the right, the \( z \) corresponds to \( z^k \), and \( (4 + a_i) \) corresponds to \( a_i^k \).