5.6 HYPERGEOMETRIC TRANSFORMATIONS 223

We can also find cases where (5.110) gives binomial sums when $z = -1$, but these are really weird. If we set $a = \frac{1}{6} - \frac{n}{3}$ and $b = -n$, we get the monstrous formula

$$
{\binom{-1}{-1} \frac{-n}{3} - \frac{n}{3} | -1} = \frac{\binom{-1}{-1} \frac{-n}{3} - \frac{n}{3}}{-8}
$$

These hypergeometrics are nondegenerate polynomials when $n \not\equiv 2 \pmod{3}$; and the parameters have been cleverly chosen so that the left-hand side can be evaluated by (5.94). We are therefore led to a truly mind-boggling result,

$$
\sum_k \binom{n}{k} \left( \frac{1}{3} \binom{\frac{1}{2} - \frac{1}{6}}{\frac{1}{2} - \frac{1}{3}} \right)^k \binom{\frac{1}{2} - \frac{1}{2}}{\frac{1}{2} - \frac{1}{3}} = \binom{2n}{n} \binom{\frac{3}{2}n - \frac{1}{2}}{\frac{3}{2}n - \frac{3}{2}} = \binom{\frac{3}{2}n - \frac{1}{2}}{\frac{3}{2}n - \frac{3}{2}}, \quad \text{integer } n \geq 0, \ n \not\equiv 2 \pmod{3}. \tag{5.113}
$$

This is the most startling identity in binomial coefficients that we’ve seen. Small cases of the identity aren’t even easy to check by hand. (It turns out that both sides do give $\frac{81}{7}$ when $n = 3$.) But the identity is completely useless, of course; surely it will never arise in a practical problem.

So that’s our hype for hypergeometrics. We’ve seen that hypergeometric series provide a high-level way to understand what’s going on in binomial coefficient sums. A great deal of additional information can be found in the classic book by Wilfred N. Bailey [15] and its sequel by Lucy Joan Slater [269].

5.7 PARTIAL HYPERGEOMETRIC SUMS

Most of the sums we’ve evaluated in this chapter range over all indices $k \geq 0$, but sometimes we’ve been able to find a closed form that works over a general range $0 \leq k < m$. For example, we know from (5.16) that

$$
\sum_{k<m} \binom{n}{k} (-1)^k = (-1)^{m-1} \binom{n-1}{m-1}, \quad \text{integer } m. \tag{5.114}
$$

The theory in Chapter 2 gives us a nice way to understand formulas like this: If $f(k) = Ag(k) = g(k + 1) - g(k)$, then we’ve agreed to write $\sum f(k) \delta k = g(k) + C$, and

$$
\sum_{a}^{b} f(k) \delta k = g(b) - g(a).
$$

Further, when $a$ and $b$ are integers with $a \leq b$, we have

$$
\sum_{a \leq k < b} f(k) \delta k = \sum_{a \leq k < b} f(k) = g(b) - g(a).
$$