Hence \( c_n = 0 \) and therefore
\[
c_1 e^{r_1 t} + \cdots + c_{n-1} e^{r_{n-1} t} = 0.
\]

(d) Repeat the preceding argument to show that \( c_{n-1} = 0 \). In a similar way it follows that \( c_{n-2} = \cdots = c_{1} = 0 \). Thus the functions \( e^{r_1 t}, \ldots, e^{r_n t} \) are linearly independent.

4.3 The Method of Undetermined Coefficients

A particular solution \( Y \) of the nonhomogeneous \( n \)th order linear equation with constant coefficients
\[
L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(t)
\]
can be obtained by the method of undetermined coefficients, provided that \( g(t) \) is of an appropriate form. While the method of undetermined coefficients is not as general as the method of variation of parameters described in the next section, it is usually much easier to use when applicable.

Just as for the second order linear equation, when the constant coefficient linear differential operator \( L \) is applied to a polynomial \( A_0 t^m + A_1 t^{m-1} + \cdots + A_m \), an exponential function \( e^{\alpha t} \), a sine function \( \sin \beta t \), or a cosine function \( \cos \beta t \), the result is a polynomial, an exponential function, or a linear combination of sine and cosine functions, respectively. Hence, if \( g(t) \) is a sum of polynomials, exponentials, sines, and cosines, or products of such functions, we can expect that it is possible to find \( Y(t) \) by choosing a suitable combination of polynomials, exponentials, and so forth, multiplied by a number of undetermined constants. The constants are then determined so that Eq. (1) is satisfied.

The main difference in using this method for higher order equations stems from the fact that roots of the characteristic polynomial equation may have multiplicity greater than 2. Consequently, terms proposed for the nonhomogeneous part of the solution may need to be multiplied by higher powers of \( t \) to make them different from terms in the solution of the corresponding homogeneous equation.

**Example 1**

Find the general solution of
\[
y''' - 3y'' + 3y' - y = 4e^t.\tag{2}
\]

The characteristic polynomial for the homogeneous equation corresponding to Eq. (2) is
\[
r^3 - 3r^2 + 3r - 1 = (r - 1)^3,
\]
so the general solution of the homogeneous equation is
\[
y_c(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t.\tag{3}
\]

To find a particular solution \( Y(t) \) of Eq. (2), we start by assuming that \( Y(t) = Ae^t \). However, since \( e^t, te^t, \) and \( t^2 e^t \) are all solutions of the homogeneous equation, we must multiply this initial choice by \( t^3 \). Thus our final assumption is that \( Y(t) = At^3 e^t, \)