negative integer. What if it is? Let’s set \( n = -N \), where \( N \) is positive. Then the term ratio for \( \sum \binom{k}{\delta k} \) is

\[
\frac{t(k+1)}{t(k)} = \frac{-(k+N)}{p(k+1)} \frac{q(k)}{p(k)} \frac{r(k)}{(k+1)}
\]

and it should be represented by \( p(k) = (k+1)^{N-1} \), \( q(k) = -1 \), \( r(k) = 1 \). Gosper’s method now tells us to look for a polynomial \( s(k) \) of degree \( d = N - 1 \); maybe there’s hope after all. For example, when \( N = 2 \) we want to solve

\[
k+1 = -((k+1)\alpha_1 + \alpha_0) = (k\alpha_1 + \alpha_0).
\]

Equating coefficients of \( k \) and 1 tells us that

\[
1 = -\alpha_1 = \alpha_1; \quad 1 = -\alpha_0 - \alpha_0;
\]

hence \( s(k) = -\frac{k}{2} \cdot \frac{1}{2} \) is a solution, and

\[
s(k) = \frac{k+1}{4} \left( \frac{-\frac{1}{2}k - \frac{1}{2}}{2} \right) = (-1)^{k-1} \frac{2k+1}{4}.
\]

Can this be the desired sum? Yes, it checks out:

\[
(-1)^k \frac{2k+3}{4} - (-1)^k \frac{12k+1}{4} = (-1)^k (k+1) = \binom{2}{k}.
\]

We can write the summation formula in another form,

\[
\sum_{k=m}^{k} \binom{-2}{k} = (-1)^{k-1} \frac{2k+1}{4} \binom{m}{0}
\]

\[
= (-1)^{m-1} \binom{m+1}{2} - \binom{m}{2}
\]

This representation conceals the fact that \( \binom{-2}{k} \) is summable in hypergeometric terms, because \( \binom{m/2}{2} \) is not a hypergeometric term.

A catalog of summable hypergeometric terms makes a useful addition to the database of hypergeometric sums mentioned earlier in this chapter. Let’s try to compile a list of the sums-in-hypergeometric-terms that we know. The geometric series \( \sum z^k \delta k \) is a very special case, which can be written\( \sum z^k \delta k = (z-1)^{-1}z^k + C \) or

\[
\sum F\left(\binom{1,1}{z}\right)_k \delta k = \frac{1}{z-1} F\left(\binom{1,1}{z}\right)_k + C.
\]

(5.124)