8 Evaluate $\sum_k \binom{n}{k} (-1)^k (1 - k/n)^n$. What is the approximate value of this sum, when $n$ is very large? Hint: This sum is $\Delta^n f (0)$ for some function $f$.

9 Show that the generalized exponentials of (5.58) obey the law

$$\mathcal{E}_t(z) = \mathcal{E}(tz)^{1/t}, \quad \text{if } t \neq 0,$$

where $\mathcal{E}(z)$ is an abbreviation for $\mathcal{E}_1(z)$.

10 Show that $-2(\ln(1 - z) + z)/z^2$ is a hypergeometric function.

11 Express the two functions

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots,$$

$$\arcsin z = z + \frac{1}{2 \cdot 3} + \frac{1 \cdot 2 \cdot 4 \cdot 5}{2 \cdot 3} + \frac{1 \cdot 2 \cdot 4 \cdot 6 \cdot 7}{2 \cdot 3} + \cdots$$

in terms of hypergeometric series.

12 Which of the following functions of $k$ is a “hypergeometric term,” in the sense of (5.115)? Explain why or why not.

a $n^k$.

b $k^n$.

c $(k! + (k+1)!)/2$.

d $H_k$, that is, $\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{k}$.

e $t(k)T(n-k)/T(n)$, when $t$ and $T$ are hypergeometric terms.

f $(t(k) + T(k))/2$, when $t$ and $T$ are hypergeometric terms.

g $(at(k) + bt(k+1) + ct(k+2))/(a + bt(1) + ct(2))$, when $t$ is a hypergeometric term.

Basics

13 Find relations between the superfactorial function $P_n = \prod_{k=1}^n k!$ of exercise 4.55, the hyperfactorial function $Q_n = \prod_{k=1}^n k^k$, and the product $R_n = \prod_{k=0}^n \binom{n}{k}$.

14 Prove identity (5.25) by negating the upper index in Vandermonde’s convolution (5.22). Then show that another negation yields (5.26).

15 What is $\sum_k \binom{n}{k}^3 (-1)^k$? Hint: See (5.29).

16 Evaluate the sum

$$\sum_k \left( \binom{2a}{a+k} \binom{2b}{b+k} \binom{2c}{c+k} \right) (-1)^k$$

when $a, b, c$ are nonnegative integers.

17 Find a simple relation between $\binom{2n-1/2}{n}$ and $\binom{2n-1/2}{2n}$.