8 Evaluate \( \sum_k \binom{n}{k} (-1)^k (1 - k/n)^n \). What is the approximate value of this sum, when \( n \) is very large? Hint: This sum is \( \Delta^n f (0) \) for some function \( f \).

9 Show that the generalized exponentials of (5.58) obey the law

\[
\mathcal{E}_t(z) = \mathcal{E}(tz)^{1/t}, \quad \text{if } t \neq 0,
\]

where \( \mathcal{E}(z) \) is an abbreviation for \( \mathcal{E}_1(z) \).

10 Show that \(-2(\ln(1 - z) + z)/z^2\) is a hypergeometric function.

11 Express the two functions

\[
\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots
\]

\[
\arcsinz = z + \frac{z^3}{2 \cdot 3} + \frac{2 \cdot 4 \cdot 5}{2 \cdot 4 \cdot 6} + \frac{2 \cdot 4 \cdot 6 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} + \ldots
\]

in terms of hypergeometric series.

12 Which of the following functions of \( k \) is a “hypergeometric term,” in the sense of (5.115)? Explain why or why not.

a \( n^k \).

b \( k^n \).

c \( (k! + (k+1)!)/2 \).

d \( H_k \), that is, \( \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{k} \).

e \( t(k)T(n - k)/T(n) \), when \( t \) and \( T \) are hypergeometric terms.

f \( (t(k) + T(k))/2 \), when \( t \) and \( T \) are hypergeometric terms.

g \( (at(k) + bt(k+1) + ct(k+2))/(a + bt(1) + ct(2)) \), when \( t \) is a hypergeometric term.

Basics

13 Find relations between the superfactorial function \( P_n = \prod_{k=1}^n k! \) of exercise 4.55, the hyperfactorial function \( Q_n = \prod_{k=1}^n k^k \), and the product \( R_n = \prod_{k=0}^n \binom{n}{k} \).

14 Prove identity (5.25) by negating the upper index in Vandermonde’s convolution (5.22). Then show that another negation yields (5.26).

15 What is \( \sum_k \binom{n}{k}^3 (-1)^k \)? Hint: See (5.29).

16 Evaluate the sum

\[
\sum_k \binom{2a}{a+k} \binom{2b}{b+k} \binom{2c}{c+k} (-1)^k
\]

when \( a, b, c \) are nonnegative integers.

17 Find a simple relation between \( \binom{2n-1/2}{n} \) and \( \binom{2n-1/2}{2n} \).