232 BINOMIAL COEFFICIENTS

18 Find an alternative form analogous to (5.35) for the product
\[
\binom{r}{k} \binom{r-1/3}{k} \binom{r-2/3}{k},
\]

19 Show that the generalized binomials of (5.58) obey the law
\[
B_t(z) = B_{t-1}(-z)^{-1}.
\]

20 Define a “generalized bloopergeometric series” by the formula
\[
G(\begin{array}{c} a_1, \ldots, a_m \\ b_1, \ldots, b_n \end{array} | z) = \sum_{k \geq 0} \frac{a_1^k \cdots a_m^k}{b_1^k \cdots b_n^k} k!,
\]
using falling powers instead of the rising ones in (5.76). Explain how \( G \) is related to \( F \).

21 Show that Euler’s definition of factorials is consistent with the ordinary definition, by showing that the limit in (5.83) is \( 1/(m-1) \cdots (1) \) when \( z = m \) is a positive integer.

22 Use (5.83) to prove the factorial duplication formula:
\[
(x!)! (x - \frac{1}{2})! = (2x)! \left( -\frac{1}{2} \right)! / 2^{2x}.
\]

23 What is the value of \( F(-n, 1; 1) \)?

24 Find \( \sum_k \binom{n}{m+k} 4^k \) by using hypergeometric series.

25 Show that
\[
(a_1 + b_1) F(\begin{array}{c} a_1, a_2, \ldots, a_m \\ b_1 + 1, b_2, \ldots, b_n \end{array} | z) = a_1 F(\begin{array}{c} a_1 + 1, a_2, \ldots, a_m \\ b_1 + 1, b_2, \ldots, b_n \end{array} | z) - b_1 F(\begin{array}{c} a_1, a_2, \ldots, a_m \\ b_1, b_2, \ldots, b_n \end{array} | z).
\]

26 Express the function \( G(z) \) in the formula
\[
F(\begin{array}{c} a_1, \ldots, a_m \\ b_1, \ldots, b_n \end{array} | z) = 1 + G(z)
\]
as a multiple of a hypergeometric series.