37 Show that an analog of the binomial theorem holds for factorial powers. That is, prove the identities
\[(x + y)^n = \sum_k \binom{n}{k} x^k y^{n-k},\]
\[(x + y)^{\bar{n}} = \sum_k \binom{n}{k} x^k y^{\bar{n-k}},\]
for all nonnegative integers n.

38 Show that all nonnegative integers n can be represented uniquely in the form \(n = (a) + (b) + (c)\) where a, b, and c are integers with \(0 \leq a < b < c\). (This is called the binomial number system.)

39 Show that if \(xy = ax + b\) then \(x^n y^n = \sum_{k=1}^{\infty} \binom{2n-1-k}{n-1} (a^k b^{n-k} x^k + a^{n-k} b^k y^k)\) for all \(n > 0\). Find a similar formula for the more general product \(x^m y^n\).

40 Find a closed form for
\[\sum_{j=1}^{m} (-1)^{j+1} \binom{r}{j} \sum_{k=1}^{n} \binom{-j + rk + s}{m - j},\]
where \(m, n \geq 0\).

41 Evaluate \(\sum_k \binom{n}{k} k! / (n + 1 + k)!\) when n is a nonnegative integer.

42 Find the indefinite sum \(\sum \binom{r+j}{j} \binom{n}{j} x^j\), and use it to compute the sum \(\sum_{k=0}^{n} (-1)^k \binom{n}{k}\) in closed form.

43 Prove the triple-binomial identity (5.28). Hint: First replace \(\binom{r+k}{m+n}\) by \(\binom{r}{m+n-j}\).

44 Use identity (5.32) to find closed forms for the double sums
\[\sum_{j,k \geq 0} (-1)^{j+k} \binom{j+k}{j} \binom{a}{j} \binom{b}{k} \binom{m+n-j-k}{m-j},\]
and
\[\sum_{j,k \geq 0} (-1)^{j+k} \binom{a}{j} \binom{m}{j} \binom{b}{k} \binom{n}{j+k} \binom{m+n}{j+k},\]
given integers \(m \geq a \geq 0\) and \(n \geq b \geq 0\).

45 Find a closed form for \(\sum_{k \leq n} \binom{2k}{k} 4^{-k}\).

46 Evaluate the following sum in closed form, when n is a positive integer:
\[\sum_k \binom{2k-1}{k} \binom{4n-2k-1}{2n-k} (-1)^{k-1} \frac{1}{(2k-1)(4n-2k-1)}\]

Hint: Generating functions win again.