47 The sum \( \sum_k \binom{r+k}{k} \binom{r+n-r-k}{n-k} \) is a polynomial in \( r \) and \( s \). Show that it doesn’t depend on \( s \).

48 The identity \( \sum_{k \leq n} \binom{n+k}{n} \cdot 2^{-k} = 2^n \) can be combined with \( \sum_{k \geq 0} \binom{n+k}{n} \cdot z^k = \frac{1}{1-z}^{n+1} \) to yield \( \sum_{k \geq n} \binom{n+k}{n} \cdot 2^{-k} = 2^n \). What is the hypergeometric form of the latter identity?

49 Use the hypergeometric method to evaluate

\[
\sum_k (-1)^k \binom{x}{k} \left( \frac{x+n-k}{n-k} \right) \frac{y}{y+n-k}.
\]

50 Prove Pfaff’s reflection law (5.101) by comparing the coefficients of \( z^n \) on both sides of the equation.

51 The derivation of (5.104) shows that

\[
\lim_{\epsilon \to 0} F(-m, -2m + 1 + \epsilon; -2m + \epsilon; 2) = 1/\binom{1/2}{m}.
\]

In this exercise we will see that slightly different limiting processes lead to distinctly different answers for the degenerate hypergeometric series \( F(-m, -2m + 1; -2m; 2) \).

52 Prove that if \( N \) is a nonnegative integer,

\[
b_1^{N} \ldots b_n^{N} \binom{a_1, \ldots, a_m, -N}{b_1, \ldots, b_n} \binom{1-b_1-N, \ldots, 1-b_n-N, -N}{1-a_1-N, \ldots, 1-a_m-N} \frac{(-1)^{m+n}}{z}.
\]

53 If we put \( b = -\frac{1}{2} \) and \( z = 1 \) in Gauss’s identity (5.110), the left side reduces to \( -1 \) while the right side is \(+1\). Why doesn’t this prove that \( -1 = +1 \)?

54 Explain how the right-hand side of (5.112) was obtained.

55 If the hypergeometric terms \( t(k) = F(a_1, \ldots, a_m; b_1, \ldots, b_n; z)k \) and \( T(k) = F(A_1, \ldots, A_M; B_1, \ldots, B_N; z)k \) satisfy \( t(k) = c(T(k+1) - T(k)) \) for all \( k \geq 0 \), show that \( z = Z \) and \( m - n = M - N \).

56 Find a general formula for \( \sum \binom{-3}{k} \delta_k \) using Gosper’s method. Show that \((-1)^{k+1} \left[ \frac{k+1}{2} \right] \left[ \frac{k+2}{2} \right] \) is also a solution.