77 What is the value of
\[ \sum_{0 \leq k_1, \ldots, k_m \leq n} \prod_{1 \leq i < m} \binom{k_{i+1}}{k_i}, \quad \text{if } m > 1? \]

78 Assuming that m is a positive integer, find a closed form for
\[ \sum_{k=0}^{2m^2} \binom{k \mod m}{(2k + 1) \mod (2m + 1)}. \]

79 a What is the greatest common divisor of \( \binom{2n}{1}, \binom{2n}{3}, \ldots, \binom{2n}{2n-1} \)? Hint: Consider the sum of these n numbers.

b Show that the least common multiple of \( \binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n} \) is equal to \( L(n + 1) / (n + 1) \), where \( L(n) = \text{lcm}(1, 2, \ldots, n) \).

80 Prove that \( \binom{n}{k} \leq (en/k)^k \) for all integers \( k, n \geq 0 \).

81 If \( 0 < \theta < 1 \) and \( 0 \leq x \leq 1 \), and if \( l, m, n \) are nonnegative integers with \( m < n \), prove the inequality
\[ (-1)^{n-m-1} \sum_k \binom{\binom{m+\theta}{k}}{\binom{n+k}{k}} x^k > 0. \]

Hint: Consider taking the derivative with respect to x.

**Bonus problems**

82 Prove that Pascal’s triangle has an even more surprising hexagon property than the one cited in the text:
\[ \gcd\left(\binom{n-1}{k-1}, \binom{n}{k}, \binom{n+1}{k+1}\right) = \gcd\left(\binom{n-1}{k}, \binom{n+1}{k+1}, \binom{n}{k-1}\right), \]
if \( 0 < k < n \). For example, \( \gcd(56, 36, 210) = \gcd(28, 120, 126) = 2 \).

83 Prove the amazing identity \( (5, 32) \) by first showing that it’s true whenever the right-hand side is zero.

84 Show that the second pair of convolution formulas, \( (5.61) \), follows from the first pair, \( (5.60) \). Hint: Differentiate with respect to \( z \).

85 Prove that
\[ \sum_{m=1}^{n} (-1)^m \sum_{1 \leq k_1 < k_2 < \ldots < k_m \leq n} \binom{k_1^3 + k_2^3 + \cdots + k_m^3 + 2^n}{n} = (-1)^n n!^3 \frac{2^n}{n}, \]
(The left side is a sum of \( 2^n - 1 \) terms.) Hint: Much more is true