Now that we have formally obtained two series solutions of Eq. (4), we can test them for convergence. Using the ratio test, it is easy to show that each of the series in Eq. (11) converges for all \( x \), and this justifies retroactively all the steps used in obtaining the solutions. Indeed, we recognize that the first series in Eq. (11) is exactly the Taylor series for \( \cos x \) about \( x = 0 \) and the second is the Taylor series for \( \sin x \) about \( x = 0 \). Thus, as expected, we obtain the solution \( y = a_0 \cos x + a_1 \sin x \).

Notice that no conditions are imposed on \( a_0 \) and \( a_1 \); hence they are arbitrary. From Eqs. (5) and (6) we see that \( y \) and \( y' \) evaluated at \( x = 0 \) are \( a_0 \) and \( a_1 \), respectively. Since the initial conditions \( y(0) \) and \( y'(0) \) can be chosen arbitrarily, it follows that \( a_0 \) and \( a_1 \) should be arbitrary until specific initial conditions are stated.

Figures 5.2.1 and 5.2.2 show how the partial sums of the series in Eq. (11) approximate \( \cos x \) and \( \sin x \). As the number of terms increases, the interval over which the approximation is satisfactory becomes longer, and for each \( x \) in this interval the accuracy of the approximation improves. However, you should always remember that a truncated power series provides only a local approximation of the solution in a neighborhood of the initial point \( x = 0 \); it cannot adequately represent the solution for large \( |x| \).

\[
\frac{1}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.
\]

\[
\begin{align*}
&= a_0 \left[ 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^n}{(2n)!} x^{2n} + \cdots \right] \\
&+ a_1 \left[ x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^n}{(2n+1)!} x^{2n+1} + \cdots \right] \\
&= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.
\end{align*}
\]