cycle arrangement obviously defines a permutation if we reverse the construction, and this one-to-one correspondence shows that permutations and cycle arrangements are essentially the same thing.

Therefore \( [n \atop k] \) is the number of permutations of \( n \) objects that contain exactly \( k \) cycles. If we sum \( \sum_{k=0}^{n} [n \atop k] \) over all \( k \), we must get the total number of permutations:

\[
\sum_{k=0}^{n} [n \atop k] = n!, \quad \text{integer } n \geq 0.
\] (6.9)

For example, \( 6 + 11 + 6 + 1 = 24 = 4! \).

Stirling numbers are useful because the recurrence relations (6.3) and (6.8) arise in a variety of problems. For example, if we want to represent ordinary powers \( x^n \) by falling powers \( x^n \), we find that the first few cases are

\[
\begin{align*}
x^0 &= x^0; \\
x^1 &= x^1; \\
x^2 &= x^2 + x^1; \\
x^3 &= x^3 + 3x^2 + x^1; \\
x^4 &= x^4 + 6x^3 + 7x^2 + x^1.
\end{align*}
\]

These coefficients look suspiciously like the numbers in Table 244, reflected between left and right; therefore we can be pretty confident that the general formula is

\[
x^n = \sum_{k=0}^{n} \left\{ n \atop k \right\} x^k, \quad \text{integer } n \geq 0.
\] (6.10)

And sure enough, a simple proof by induction clinches the argument: We have \( x \cdot x^k = x^{k+1} + kx^k \) because \( x^{k+1} = x^k(x - k) \); hence \( x \cdot x^{n-1} \) is

\[
x \sum_{k} \left\{ n - 1 \atop k \right\} x^k = \sum_{k} \left\{ n - 1 \atop k \right\} x^{k+1} + \sum_{k} \left\{ n - 1 \atop k \right\} kx^k
\]

\[
= \sum_{k} \left\{ n - 1 \atop k - 1 \right\} x^k + \sum_{k} \left\{ n - 1 \atop k \right\} kx^k
\]

\[
= \sum_{k} \left( k \left\{ n - 1 \atop k \right\} + \left\{ n - 1 \atop k - 1 \right\} \right) x^k = \sum_{k} \left\{ n \atop k \right\} x^k.
\]

In other words, Stirling subset numbers are the coefficients of factorial powers that yield ordinary powers.