We can go the other way too, because Stirling cycle numbers are the coefficients of ordinary powers that yield factorial powers:

\[
\begin{align*}
\chi_0^T &= x^0; \\
\chi_1^T &= x^1; \\
\chi_2^T &= x^2 + x^1; \\
\chi_3^T &= x^3 + 3x^2 + 2x^1; \\
\chi_4^T &= x^4 + 6x^3 + 11x^2 + 6x^1.
\end{align*}
\]

We have \((x+n-1)x^k = x^{k+1} + (n-1)x^k\), so a proof like the one just given shows that

\[
(x + n - 1)x^{n-k} = (x + n - 1) \sum_k \left[ \frac{n-1}{k} \right] x^k = \sum_k \left[ \frac{n}{k} \right] x^k.
\]

This leads to a proof by induction of the general formula

\[
\sum_k \left[ \frac{n}{k} \right] x^k, \quad \text{integer } n \geq 0. \tag{6.11}
\]

(Setting \(x = 1\) gives (6.9) again.)

But wait, you say. This equation involves rising factorial powers \(\chi^R\), while (6.10) involves falling factorials \(\chi^F\). What if we want to express \(\chi^R\) in terms of ordinary powers, or if we want to express \(\chi^F\) in terms of rising powers? Easy; we just throw in some minus signs and get

\[
\begin{align*}
\chi^F &= \sum_k \left\{ \frac{n}{k} \right\} (-1)^{n-k} \chi^F, \quad \text{integer } n \geq 0; \tag{6.12} \\
\chi^R &= \sum_k \left[ \frac{n}{k} \right] (-1)^{n-k} \chi^F, \quad \text{integer } n \geq 0. \tag{6.13}
\end{align*}
\]

This works because, for example, the formula

\[
\chi_4^F = x(x-1)(x-2)(x-3) = x^4 - 6x^3 + 11x^2 - 6x
\]

is just like the formula

\[
\chi_4^R = x(x+1)(x+2)(x+3) = x^4 + 6x^3 + 11x^2 + 6x
\]

but with alternating signs. The general identity

\[
\chi^F = (-1)^n (-x)^R \tag{6.14}
\]

of exercise 2.17 converts (6.10) to (6.12) and (6.11) to (6.13) if we negate \(x\).