We can go the other way too, because Stirling cycle numbers are the coefficients of ordinary powers that yield factorial powers:

\[ \begin{align*}
  x^0 &= x^0; \\
  x^1 &= x^1; \\
  x^2 &= x^2 + x^1; \\
  x^3 &= x^3 + 3x^2 + 2x^1; \\
  x^4 &= x^4 + 6x^3 + 11x^2 + 6x^1. 
\end{align*} \]

We have \((x + n - 1)x^k = x^{k+1} + (n - 1)x^k\), so a proof like the one just given shows that

\[ (x + n - 1)x^n = (x + n - 1) \sum_k \binom{n-1}{k} x^k = \sum_k \binom{n}{k} x^k. \]

This leads to a proof by induction of the general formula

\[ x^n = \sum_k \binom{n}{k} x^k, \quad \text{integer } n \geq 0. \tag{6.11} \]

(Setting \(x = 1\) gives (6.9) again.)

But wait, you say. This equation involves rising factorial powers \(x^\uparrow\), while (6.10) involves falling factorials \(x^\downarrow\). What if we want to express \(x^n\) in terms of ordinary powers, or if we want to express \(x^\downarrow\) in terms of rising powers? Easy; we just throw in some minus signs and get

\[ \begin{align*}
  x^n &= \sum_k \binom{n}{k} (-1)^{n-k} x^\downarrow, \quad \text{integer } n \geq 0; \tag{6.12} \\
  x^\downarrow &= \sum_k \binom{n}{k} (-1)^{n-k} x^k, \quad \text{integer } n \geq 0. \tag{6.13} 
\end{align*} \]

This works because, for example, the formula

\[ x^4 = x(x - 1)(x - 2)(x - 3) = x^4 - 6x^3 + 11x^2 - 6x \]

is just like the formula

\[ x^\uparrow = x(x + 1)(x + 2)(x + 3) = x^4 + 6x^3 + 11x^2 + 6x \]

but with alternating signs. The general identity

\[ x^n = (-1)^n (-x)^\downarrow \tag{6.14} \]

of exercise 2.17 converts (6.10) to (6.12) and (6.11) to (6.13) if we negate \(x\).