such as the ratio test. However, even without knowing the formula for $a_n$, we shall see in Section 5.3 that it is possible to establish that the series in Eq. (20) converge for all $x$, and further define functions $y_3$ and $y_4$ that are linearly independent solutions of the Airy equation (12). Thus

$$y = a_0 y_3(x) + a_1 y_4(x)$$

is the general solution of Airy’s equation for $-\infty < x < \infty$.

It is worth emphasizing, as we saw in Example 3, that if we look for a solution of Eq. (1) of the form $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$, then the coefficients $P(x)$, $Q(x)$, and $R(x)$ in Eq. (1) must also be expressed in powers of $x - x_0$. Alternatively, we can make the change of variable $x - x_0 = t$, obtaining a new differential equation for $y$ as a function of $t$, and then look for solutions of this new equation of the form $\sum_{n=0}^{\infty} a_n t^n$. When we have finished the calculations, we replace $t$ by $x - x_0$ (see Problem 19).

In Examples 2 and 3 we have found two sets of solutions of Airy’s equation. The functions $y_1$ and $y_2$ defined by the series in Eq. (17) are linearly independent solutions of Eq. (12) for all $x$, and this is also true for the functions $y_3$ and $y_4$ defined by the series in Eq. (20). According to the general theory of second order linear equations each of the first two functions can be expressed as a linear combination of the latter two functions and vice versa—a result that is certainly not obvious from an examination of the series alone.

Finally, we emphasize that it is not particularly important if, as in Example 3, we are unable to determine the general coefficient $a_n$ in terms of $a_0$ and $a_1$. What is essential is that we can determine as many coefficients as we want. Thus we can find as many terms in the two series solutions as we want, even if we cannot determine the general term. While the task of calculating several coefficients in a power series solution is not difficult, it can be tedious. A symbolic manipulation package can be very helpful here; some are able to find a specified number of terms in a power series solution in response to a single command. With a suitable graphics package one can also produce plots such as those shown in the figures in this section.

### PROBLEMS

In each of Problems 1 through 14 solve the given differential equation by means of a power series about the given point $x_0$. Find the recurrence relation; also find the first four terms in each of two linearly independent solutions (unless the series terminates sooner). If possible, find the general term in each solution.

1. $y'' - y = 0$, $x_0 = 0$
2. $y'' - xy' - y = 0$, $x_0 = 0$
3. $y'' - xy' - y = 0$, $x_0 = 1$
4. $y'' + k^2x^2y = 0$, $x_0 = 0$, $k$ a constant
5. $(1 - x)y'' + y = 0$, $x_0 = 0$
6. $(2 + x^2)y'' + xy' + 4y = 0$, $x_0 = 0$
7. $y'' + xy + 2y = 0$, $x_0 = 0$
8. $xy'' + y' + xy = 0$, $x_0 = 1$
9. $(1 + x^2)y'' - 4xy' + 6y = 0$, $x_0 = 0$
10. $(4 - x^2)y'' + 2y = 0$, $x_0 = 0$
11. $(3 - x^2)y'' - 3xy' - y = 0$, $x_0 = 0$
12. $(1 - x)y'' + xy' - y = 0$, $x_0 = 0$
13. $2y'' + xy' + 3y = 0$, $x_0 = 0$
14. $2y'' + (x + 1)y' + 3y = 0$, $x_0 = 2$