We can remember when to stick the \((-1)^{n-k}\) factor into a formula like (6.12) because there's a natural ordering of powers when \(x\) is large:

\[
x^0 > x^n > x^3, \quad \text{for all } x > n > 1.
\]

(6.30)

The Stirling numbers \([n]\) and \({n\choose k}\) are nonnegative, so we have to use minus signs when expanding a “small” power in terms of “large” ones.

We can plug (6.11) into (6.12) and get a double sum:

\[
x^n = \sum_k \left( \begin{array}{c} n \\ k \end{array} \right) (-1)^{n-k} x^k = \sum_{k,m} \left( \begin{array}{c} n \\ k \end{array} \right) \left[ \begin{array}{c} k \\ m \end{array} \right] (-1)^{n-k} x^m.
\]

This holds for all \(x\), so the coefficients of \(x^0, x^1, \ldots, x^{n-1}, x^{n+1}, x^{n+2}, \ldots\) on the right must all be zero and we must have the identity

\[
\sum_k \left( \begin{array}{c} n \\ k \end{array} \right) \left[ \begin{array}{c} k \\ m \end{array} \right] (-1)^{n-k} = [m=n], \quad \text{integers } m, n \geq 0.
\]

(6.31)

Stirling numbers, like binomial coefficients, satisfy many surprising identities. But these identities aren’t as versatile as the ones we had in Chapter 5, so they aren’t applied nearly as often. Therefore it’s best for us just to list the simplest ones, for future reference when a tough Stirling nut needs to be cracked. Tables 250 and 251 contain the formulas that are most frequently useful; the principal identities we have already derived are repeated there.

When we studied binomial coefficients in Chapter 5, we found that it was advantageous to define \({n\choose k}\) for negative \(n\) in such a way that the identity \({n\choose k} = \binom{n-1}{k} + \binom{n-1}{k-1}\) is valid without any restrictions. Using that identity to extend the \({n\choose k}\)'s beyond those with combinatorial significance, we discovered (in Table 164) that Pascal’s triangle essentially reproduces itself in a rotated form when we extend it upward. Let’s try the same thing with Stirling’s triangles: What happens if we decide that the basic recurrences

\[
\left( \begin{array}{c} n \\ k \end{array} \right) = k \left( \begin{array}{c} n-1 \\ k \end{array} \right) + \left( \begin{array}{c} n-1 \\ k-1 \end{array} \right)
\]

\[
\left[ \begin{array}{c} n \\ k \end{array} \right] = (n-1) \left[ \begin{array}{c} n-1 \\ k \end{array} \right] + \left[ \begin{array}{c} n-1 \\ k-1 \end{array} \right]
\]

are valid for all integers \(n\) and \(k\)? The solution becomes unique if we make the reasonable additional stipulations that

\[
\left( \begin{array}{c} 0 \\ k \end{array} \right) = [k=0] \quad \text{and} \quad \left( \begin{array}{c} n \\ 0 \end{array} \right) = \left[ \begin{array}{c} n \\ 0 \end{array} \right] = [n=0].
\]

(6.32)