commutes, where the horizontal and vertical arrows are the canonical injection and projection respectively.

Thus we may speak unambiguously of $\omega_\lambda(g)$, for $g$ belonging to $\text{Sp}_F$.

4. The $\Theta$-distribution. $F$ continues as in §3.

**Theorem 4.1.** There is a linear functional $\Theta$ on (the space $Y$ of) $\rho_\lambda^\infty$ which is invariant by $H_F$. That is, $\Theta(\rho_\lambda^\infty(h)(y)) = \Theta(y)$ for $h \in H_F$ and $y \in Y$. The functional $\Theta$ is unique up to multiples.

This result is essentially a strong form of the Poisson Summation Formula. Results of the same type are in [C 1]. Existence of $\Theta$ is implicit in Weil [Wi 1], and exploited classically. Existence of $\Theta$ is more or less responsible for the lifting $\beta$ of §3.

Since $\text{Sp}_F$ normalizes $H_F$, we see by formula (2.1) and the uniqueness of $\Theta$ that $\omega_\lambda(\text{Sp}_F)$ preserves $\Theta$ up to multiples. Since $\text{Sp}_F$ is its own commutator subgroup, $\theta$ will in fact be invariant by $\text{Sp}_F$. Therefore, if $u$ is a vector in the space of $\rho_\lambda^\infty$, the function on $\tilde{\text{Sp}}_A$ defined by $\theta(u)(g) = \theta(\omega_\lambda(g)u)$, $g \in \tilde{\text{Sp}}_A$, actually factors to the coset space $X = \text{Sp}_F \backslash \tilde{\text{Sp}}_A$. It may also be computed that $\theta(u)(g)$ is of “moderate growth” at $\infty$ on $X$, so that actually $\theta(u)$ is an automorphic form on $\tilde{\text{Sp}}_A$. We call $\theta(u)$ the theta-series attached to $u$.

If dim $W = 2$ and $u$ is chosen appropriately, then $\theta(u)$ will be the classical theta-series of Jacobi. With somewhat more general $u$, but still a restricted class, one will obtain “theta-constants with characteristic” [I]. Thus the $\theta$-distribution gives rise to automorphic forms which generalize well-known $\theta$-series. However, the functions $\theta(u)$ are still very special automorphic forms. But if $G \subseteq \text{Sp}_F$ is a reductive algebraic subgroup, then we may restrict $\theta(u)$ to $\tilde{G}_A$ (where this is the inverse image in $\tilde{\text{Sp}}_A$ of $G_A \subseteq \text{Sp}_A$) and thereby obtain automorphic forms on $\tilde{G}_A$. If $G$ is a relatively small subgroup, this will produce fairly general automorphic forms on $\tilde{G}_A$. For general $G$ it will be difficult to be very precise about the nature of $\theta(u)|\tilde{G}_A$, but if $G$ belongs to a reductive dual pair as defined below then the structure of the $\theta(u)|\tilde{G}_A$ should be specifiable in considerable detail. It is in this circumstance that the relevance of $\theta$-series to the theory of automorphic forms lies.

5. Reductive dual pairs. Here $F$ is arbitrary (except still not of characteristic 2).

Let $(G, G')$ be a pair of subgroups of Sp. We say $(G, G')$ form a reductive dual pair if

(i) $G$ and $G'$ act absolutely reductively on $W$; and

(ii) $G$ is the centralizer of $G'$ in Sp and vice versa.

Reductive dual pairs may be classified as follows. If $(G, G')$ is a reductive dual pair in Sp, and if $W = W_1 \oplus W_2$ is an orthogonal direct sum decomposition where $W_1$ and $W_2$ are invariant by $G \cdot G'$, then we say $(G, G')$ is reducible. The restrictions $(G_i, G'_i)$ of $(G, G')$ to the $W_i$ define reductive dual pairs in the $\text{Sp}(W_i)$. We say that
(G, G') is the direct sum of the (G_i, G_j). If (G, G') is not reducible, it is irreducible. Any pair is a direct sum in an essentially unique way of irreducible pairs. The irreducible pairs may be described as follows.

Type II. There are maximal isotropic subspaces X, Y ⊆ W with X ⊕ Y = W and X and Y invariant by G · G'. In this case there exist:
(a) a division algebra D over F (not necessarily central);
(b) a right D-module X_1 and a left D-module X_2; such that
(c) X ∼ X_1 ⊗_D X_2 in such fashion that (G, G') are identified to (GL_D(X_1), GL_D(X_2)).

Type I. The joint action of G · G' is irreducible on V. Then there exist
(a) a division algebra D,
(b) with involution h, and
(c) D-modules V_1 and V_2
(d) with forms ( , )_1 and ( , )_2, one h-Hermitian and the other h-skew-Hermitian; such that
(e) W ∼ V_1 ⊗ V_2 in such a way that (G, G') are identified to the isometry groups of ( , )_1 and ( , )_2, and
(f) ( , ) = tr_{D/F}(( , )_1 ⊗ ( , )_2). Here tr_{D/F} is reduced trace. The algebra D is not necessarily central over F. Although the tensor product of forms over D does not make sense, when you take traces you get a good F-bilinear form.

Remarks. (a) Essentially the same classification can be found in Weil [Wi 2] and elsewhere. It essentially goes back to Albert. It may also be deduced from the results in [Sa].
(b) Over a local field F a division algebra with involution is either a quadratic extension or a quaternion algebra over some extension field of F.
(c) Consider a pair (G, G') over a number field F. Suppose (G, G') is irreducible of type I, that h is a positive involution on D, that V_2 ∼ D with h-Hermitian form (x, y)_2 = x^h y. Then V_1 ∼ W has a h-skew-Hermitian form ( , )_1. The data D, h, V_1 and ( , )_2 are 4 of the 7 things Shimura [Sh] uses to specify his PEL-types. The rest of Shimura’s data arises further on in the development of the theory of reductive dual pairs. Thus Shimura’s PEL-types arise naturally in the context of reductive dual pairs. It would be an interesting study to see how much of the work of Shimura and others on these objects can be understood in terms of the oscillator representation.

6. Local duality. F is now again a local field.
Let H ⊆ Sp be a closed subgroup. Define
\[ \mathcal{R}(H) = \{ \sigma : \sigma \text{ is an irreducible smooth representation of } H, \]
\[ \text{and there exists a nontrivial } H\text{-intertwining map } \alpha : \omega^\infty \rightarrow \sigma \}. \]

As before, if G ⊆ Sp, then G̃ is the inverse image of G in Sp. If (G, G') ⊆ Sp is a reductive dual pair, then G̃ and G̃' commute in Sp, and so an irreducible admissible representation of G · G' will pull back to an irreducible admissible representation of G · G'. Such a representation has the form σ ⊗ σ' where σ and σ' are irreducible admissible representations of G̃ and G̃' respectively. Thus \( \mathcal{R}(G · G') \) defines (is the graph of) a correspondence between representations of G̃ and representations of G̃'.