According to Theorem 5.3.1 the series solutions of the Airy equation in Examples 2 and 3 of the preceding section converge for all values of $x$ and $x - 1$, respectively, since in each problem $P(x) = 1$ and hence is never zero.

A series solution may converge for a wider range of $x$ than indicated by Theorem 5.3.1, so the theorem actually gives only a lower bound on the radius of convergence of the series solution. This is illustrated by the Legendre polynomial solution of the Legendre equation given in the next example.

**Example 3**

Determine a lower bound for the radius of convergence of series solutions about $x = 0$ for the Legendre equation

$$\left(1 - x^2\right)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$$

where $\alpha$ is a constant.

Note that $P(x) = 1 - x^2$, $Q(x) = -2x$, and $R(x) = \alpha(\alpha + 1)$ are polynomials, and that the zeros of $P$, namely, $x = \pm 1$, are a distance 1 from $x = 0$. Hence a series solution of the form $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < 1$ at least, and possibly for larger values of $x$. Indeed, it can be shown that if $\alpha$ is a positive integer, one of the series solutions terminates after a finite number of terms and hence converges not just for $|x| < 1$ but for all $x$. For example, if $\alpha = 1$, the polynomial solution is $y = x$.

See Problems 22 through 29 at the end of this section for a more complete discussion of the Legendre equation.

**Example 4**

Determine a lower bound for the radius of convergence of series solutions of the differential equation

$$\left(1 + x^2\right)y'' + 2xy' + 4x^2y = 0$$

about the point $x = 0$; about the point $x = -\frac{1}{2}$.

Again $P$, $Q$, and $R$ are polynomials, and $P$ has zeros at $x = \pm i$. The distance in the complex plane from 0 to $\pm i$ is 1, and from $-\frac{1}{2}$ to $\pm i$ is $\sqrt{1 + \frac{1}{4}} = \sqrt{5}/2$. Hence in the first case the series $\sum_{n=0}^{\infty} a_n x^n$ converges at least for $|x| < 1$, and in the second case the series $\sum_{n=0}^{\infty} b_n (x + \frac{1}{2})^n$ converges at least for $|x + \frac{1}{2}| < \sqrt{5}/2$.

An interesting observation that we can make about Eq. (9) follows from Theorems 3.2.1 and 5.3.1. Suppose that initial conditions $y(0) = y_0$ and $y'(0) = y'_0$ are given. Since $1 + x^2 \neq 0$ for all $x$, we know from Theorem 3.2.1 that there exists a unique solution of the initial value problem on $-\infty < x < \infty$. On the other hand, Theorem 5.3.1 only guarantees a series solution of the form $\sum_{n=0}^{\infty} a_n x^n$ (with $a_0 = y_0$, $a_1 = y'_0$) for $-1 < x < 1$. The unique solution on the interval $-\infty < x < \infty$ may not have a power series about $x = 0$ that converges for all $x$.

**Example 5**

Can we determine a series solution about $x = 0$ for the differential equation

$$y'' + \sin(x)y' + (1 + x^2)y = 0,$$

and if so, what is the radius of convergence?