Second-order Eulerian numbers are important chiefly because of their connection with Stirling numbers [119]: We have, by induction on \( n \),

\[
\begin{align*}
\left\{ \frac{x}{x - n} \right\} &= \sum_k \left\langle \binom{n}{k}\right\rangle \left( \frac{x + n - 1 - k}{2n} \right)^k, \quad \text{integer } n \geq 0; \quad (6.43) \\
\left[ \frac{x}{x - n} \right] &= \sum_k \left\langle \binom{n}{k}\right\rangle \left( \frac{x + k}{2n} \right)^k, \quad \text{integer } n \geq 0. \quad (6.44)
\end{align*}
\]

For example,

\[
\begin{align*}
\left\{ \frac{x}{x - 1} \right\} &= \left( \frac{x}{2} \right), & \left[ \frac{x}{x - 1} \right] &= \left( \frac{x}{2} \right); \\
\left\{ \frac{x}{x - 2} \right\} &= \left( \frac{x + 1}{4} \right) + 2 \left( \frac{x}{4} \right), & \left[ \frac{x}{x - 2} \right] &= \left( \frac{x}{4} \right) + 2 \left( \frac{x + 1}{4} \right); \\
\left\{ \frac{x}{x - 3} \right\} &= \left( \frac{x + 2}{6} \right) + 8 \left( \frac{x + 1}{6} \right) + 6 \left( \frac{x}{6} \right), & \left[ \frac{x}{x - 3} \right] &= \left( \frac{x}{6} \right) + 8 \left( \frac{x + 1}{6} \right) + 6 \left( \frac{x + 2}{6} \right).
\end{align*}
\]

(We already encountered the case \( n = 1 \) in (6.7).) These identities hold whenever \( x \) is an integer and \( n \) is a nonnegative integer. Since the right-hand sides are polynomials in \( x \), we can use (6.43) and (6.44) to define Stirling numbers \( \left\{ \binom{x}{n} \right\} \) and \( \left[ \binom{x}{n} \right] \) for arbitrary real (or complex) values of \( x \).

If \( n > 0 \), these polynomials \( \left\{ \binom{x}{n} \right\} \) and \( \left[ \binom{x}{n} \right] \) are zero when \( x = 0, x = 1, \ldots, \) and \( x = n \); therefore they are divisible by \( (x - 0), (x - 1), \ldots, \) and \( (x - n) \). It’s interesting to look at what’s left after these known factors are divided out. We define the Stirling polynomials \( \sigma_n(x) \) by the rule

\[
\sigma_n(x) = \left[ \frac{x^n}{x - n} \right] / (x(x - 1) \ldots (x - n)). \quad (6.45)
\]

(The degree of \( \sigma_n(x) \) is \( n - 1 \).) The first few cases are

\[
\begin{align*}
\sigma_0(x) &= 1/x; \\
\sigma_1(x) &= 1/2; \\
\sigma_2(x) &= (3x - 1)/24; \\
\sigma_3(x) &= (x^2 - x)/48; \\
\sigma_4(x) &= (15x^3 - 30x^2 + 5x + 2)/5760.
\end{align*}
\]

They can be computed via the second-order Eulerian numbers; for example,

\[
\sigma_3(x) = \frac{(x-4)(x-5) + 8(x+1)(x-4) + 6(x+2)(x+1)}{6!}.
\]