where \([n/2]\) denotes the greatest integer less than or equal to \(n/2\). By observing the form of \(P_n(x)\) for \(n\) even and \(n\) odd, show that \(P_n(-1) = (-1)^n\).

26. The Legendre polynomials play an important role in mathematical physics. For example, in solving Laplace’s equation (the potential equation) in spherical coordinates we encounter the equation

\[
\frac{d^2 F(\varphi)}{d\varphi^2} + \cot \varphi \frac{d F(\varphi)}{d\varphi} + n(n + 1)F(\varphi) = 0, \quad 0 < \varphi < \pi,
\]

where \(n\) is a positive integer. Show that the change of variable \(x = \cos \varphi\) leads to the Legendre equation with \(\alpha = n\) for \(y = f(x) = F(\arccos x)\).

27. Show that for \(n = 0, 1, 2, 3\) the corresponding Legendre polynomial is given by

\[
P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.
\]

This formula, known as Rodrigues’ (1794–1851) formula, is true for all positive integers \(n\).

28. Show that the Legendre equation can also be written as

\[
[(1 - x^2)y']' = -\alpha(\alpha + 1)y.
\]

Then it follows that \([(1 - x^2)P_n'(x)]' = -n(n + 1)P_n(x)\) and \([(1 - x^2)P_m'(x)]' = -m(m + 1)P_m(x)\). By multiplying the first equation by \(P_m(x)\) and the second equation by \(P_n(x)\), and then integrating by parts, show that

\[
\int_{-1}^{1} P_n(x) P_m(x) \, dx = 0 \quad \text{if} \quad n \neq m.
\]

This property of the Legendre polynomials is known as the orthogonality property. If \(m = n\), it can be shown that the value of the preceding integral is \(2/(2n + 1)\).

29. Given a polynomial \(f\) of degree \(n\), it is possible to express \(f\) as a linear combination of \(P_0, P_1, P_2, \ldots, P_n\):

\[
f(x) = \sum_{k=0}^{n} a_k P_k(x).
\]

Using the result of Problem 28, show that

\[
a_k = \frac{2k + 1}{2} \int_{-1}^{1} f(x) P_k(x) \, dx.
\]

5.4 Regular Singular Points

In this section we will consider the equation

\[
P(x)y'' + Q(x)y' + R(x)y = 0
\]

in the neighborhood of a singular point \(x_0\). Recall that if the functions \(P, Q,\) and \(R\) are polynomials having no common factors, the singular points of Eq. (1) are the points for which \(P(x) = 0\).