Example 6.2: Let $\alpha$ be the hyperplane $3x - 4y + z - 2w = 7$ in $\mathbb{R}^4$. The distance $d$ between the point $(1, -4, 2, 3)$ and $\alpha$ is

$$d = \frac{|3 \cdot 1 - 4 \cdot (-4) + 1 \cdot (-2) - 2 \cdot 3 - 7|}{\sqrt{3^2 + (-4)^2 + 1^2 + (-2)^2}} = \frac{4}{\sqrt{30}}$$

The distance $\rho$ between $\alpha$ and the origin is $\rho = \frac{|7|}{\sqrt{3^2 + (-4)^2 + 1^2 + (-2)^2}} = \frac{7}{\sqrt{30}}$.

**Solved Problems**

**DISTANCE IN THE PLANE**

22.1. Find the distance $d$ between (i) $(-3, 5)$ and $(6, 4)$, (ii) $(5, -2)$ and $(-1, -4)$.

In each case use the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

(i) $d = \sqrt{(6 + 3)^2 + (4 - 5)^2} = \sqrt{81 + 1} = \sqrt{82}$

(ii) $d = \sqrt{(-1 - 5)^2 + (-4 + 2)^2} = \sqrt{36 + 4} = \sqrt{40}$

22.2. Show that the points $p = (-1, 4)$, $q = (2, 1)$ and $r = (3, 5)$ are the vertices of an isosceles triangle.

Compute the distance between each pair of points:

$$d(p, q) = \sqrt{(2 + 1)^2 + (1 - 4)^2} = \sqrt{9 + 9} = \sqrt{18}$$

$$d(p, r) = \sqrt{(3 + 1)^2 + (5 - 4)^2} = \sqrt{16 + 1} = \sqrt{17}$$

$$d(q, r) = \sqrt{(3 - 2)^2 + (5 - 1)^2} = \sqrt{1 + 1} = \sqrt{17}$$

Since $d(p, r) = d(q, r)$, the triangle is isosceles.

22.3. Show that the points $p = (4, -2)$, $q = (-1, 3)$ and $r = (1, 4)$ are the vertices of a right triangle, and find its area.

Compute the distance between each pair of points:

$$d(p, q) = \sqrt{(-1 - 4)^2 + (3 + 2)^2} = \sqrt{25 + 25} = \sqrt{50}$$

$$d(p, r) = \sqrt{(1 - 4)^2 + (4 + 2)^2} = \sqrt{9 + 36} = \sqrt{45}$$

$$d(q, r) = \sqrt{(1 + 1)^2 + (4 - 3)^2} = \sqrt{4 + 1} = \sqrt{5}$$

Since $d(q, r)^2 + d(p, r)^2 = d(p, q)^2$, the triangle is a right triangle.

The area of the right triangle is $A = \frac{1}{2}(\sqrt{45})(\sqrt{5}) = \frac{\sqrt{225}}{2}$.

22.4. Find the point equidistant from the points $a = (-4, 3)$, $b = (5, 6)$ and $c = (4, -1)$.

Let $p = (x, y)$ be the required point. Then $d(p, a) = d(p, b) = d(p, c)$. Since $d(p, a) = d(p, b)$,

$$\sqrt{(x + 4)^2 + (y - 3)^2} = \sqrt{(x - 5)^2 + (y - 6)^2} \quad \text{or} \quad 3x + y = 6$$

Since $d(p, a) = d(p, c)$,

$$\sqrt{(x + 4)^2 + (y - 3)^2} = \sqrt{(x - 4)^2 + (y + 1)^2} \quad \text{or} \quad 2x - y = -1$$

Solve the two linear equations simultaneously to obtain $x = 1$ and $y = 3$. Thus $p = (1, 3)$ is the required point.