“Harmonic,” since a tone of wavelength $l/n$ is called the $n$th harmonic of a tone whose wavelength is 1. The first few values look like this:

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_n$</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>11</td>
<td>25</td>
<td>49</td>
<td>63</td>
<td>761</td>
<td>920</td>
<td>1129</td>
<td>7381</td>
</tr>
</tbody>
</table>

Exercise 21 shows that $H_n$ is never an integer when $n > 1$.

Here’s a card trick, based on an idea by R. T. Sharp [264], that illustrates how the harmonic numbers arise naturally in simple situations. Given $n$ cards and a table, we’d like to create the largest possible overhang by stacking the cards up over the table’s edge, subject to the laws of gravity:

![Card Trick Diagram]

To define the problem a bit more, we require the edges of the cards to be parallel to the edge of the table; otherwise we could increase the overhang by rotating the cards so that their corners stick out a little farther. And to make the answer simpler, we assume that each card is 2 units long.

With one card, we get maximum overhang when its center of gravity is just above the edge of the table. The center of gravity is in the middle of the card, so we can create half a cardlength, or 1 unit, of overhang.

With two cards, it’s not hard to convince ourselves that we get maximum overhang when the center of gravity of the top card is just above the edge of the second card, and the center of gravity of both cards combined is just above the edge of the table. The joint center of gravity of two cards will be in the middle of their common part, so we are able to achieve an additional half unit of overhang.

This pattern suggests a general method, where we place cards so that the center of gravity of the top $k$ cards lies just above the edge of the $k+1$st card (which supports those top $k$). The table plays the role of the $n+1$st card. To express this condition algebraically, we can let $d_k$ be the distance from the extreme edge of the top card to the corresponding edge of the $k$th card from the top. Then $d_1 = 0$, and we want to make $d_{k+1}$ the center of gravity of the first $k$ cards:

$$d_{k+1} = \frac{(d_1 + 1) + (d_2 + 1) + \cdots + (d_k + 1)}{k}, \text{ for } 1 < k < n.$$ (6.55)