We will sketch here how the conjectures and results of §§6, 7, 8 are related to invariant-theoretic considerations. For economy of space, we will only consider a non-Archimedean field. Unfortunately, this case is more than one short step removed from the algebraic classical invariant theory of [Wγy], and the strong relations may not be immediately clear. For a recasting of classical invariant theory in terms of reductive dual pairs, including a proof of the local duality for pairs \((G, G')\) over \(R\) where \(G\) or \(G'\) is compact, see [H12].

The first necessity for invariant theory is a supply of "natural" invariants. To provide these we return to the symplectic vector space \(W\) and the Heisenberg group \(H = H(W)\). Let \(\chi\) be a nontrivial character of the center of \(H\). For any subgroup \(A \subseteq H\), we define \(A^\circ\) to be the centralizer of \(A\) in \(H\) modulo the kernel of \(\chi\). If \(A = A^\circ\) we say \(A\) polarizes \(\chi\). If \(A\) polarizes \(\chi\), then \(F\) (here considered as the center of \(H\)) is contained in \(A\), and \(A = (A \cap W) \oplus F\). The set \(A \cap W\) will be a subgroup of \(W\), hence a \(\mathbb{Z}_p\)-module. If \(A \cap W\) is an \(O\)-module, \(O\) being the integers in \(F\), we will say \(A\) is an \(F\)-polarization or a polarization over \(F\). Let \(Q\) denote the set of all polarizations and \(Q_F\) the subset of \(F\)-polarizations. The action of \(Sp\) on \(H\) permutes the polarizations. If \(dim \ W = n\), then \(Q_F\) consists of \(n + 1\) orbits for \(Sp\).

If \(A \subseteq H\) polarizes \(\chi\), then there is a unique extension \(\chi_A\) of \(\chi\) to \(A\) such that \(A \cap W \subseteq \ker \chi_A\).

**Theorem 9.1.** Let the representation \(\rho^*_\omega\) be realized on a vector space \(Y\). For each polarization \(A\) of \(\chi\) there is a linear form \(\lambda_A\) on \(Y\) such that

\[
(9.1) \quad \lambda_A(\rho^*_\omega(a)y) = \chi_A(a)\lambda_A(y) \quad \text{for } a \in A, \ y \in Y.
\]

The functional \(\lambda_A\) is specified up to multiples by equation (9.1). The resulting map \(\mathcal{Z}: Q \rightarrow P(Y^*)\) embeds \(Q\) as a compact subset in \(P(Y^*)\). The map \(\mathcal{Z}\) is equivariant for the natural actions of \(Sp\) on \(Q\) and on \(P(Y^*)\).

Since \(P(Y^*)\) has a natural topology, the map \(\mathcal{Z}\) implicitly topologizes \(Q\). In this topology \(Q_F\) is a closed subset of \(Q\) and each \(Sp\) orbit in \(Q_F\) has its natural topology. We can label the \(Sp\)-orbits in \(Q_F\) by \(\{Q^F_p\}_{p=0}^p\) in such a way that \(Q^F_p\) is open and dense in \(\bigcup_{j=0}^p Q^F_p\). Thus \(Q^F_p\) is the unique closed orbit. It consists of polarizing \(A\) such that \(A \cap W\) is a maximal isotropic subspace of \(W\). Also \(Q^F_p\) is the unique open orbit. It consists of \(A\) such that \(A \cap W\) is a lattice (an open compact \(O\)-module) in \(W\).

The way \(Q_F\) is used to produce invariants is as follows. If \(G \subseteq Sp\), and \(A \in Q_F\), and \(A\) is invariant by \(G\), then \(\lambda_A\) must be invariant up to multiples by \(\omega^*_\omega(\bar{G})\). Thus there is a quasi-character \(\phi_A\) on \(\bar{G}\) such that

\[
(9.2) \quad \omega^*_\omega(\bar{g}) \lambda_A = \phi_A(\bar{g}) \lambda_A \quad \text{for } \bar{g} \in \bar{G}.
\]

Thus the existence of \(G\)-fixed points in \(Q_F\) leads to the existence of at least quasi-invariant linear forms on \(Y\), and these forms will often be invariant. The analogy with construction of automorphic forms (\(\theta\)-series) by means of the \(\theta\)-distribution as formulated in §4 is clear. This analogy will be developed further in §12. This is also essentially the basis of the First Fundamental Theorems of classical invariant theory, though the parallel is perhaps not so immediately seen.

Let \((G, G') \subseteq Sp\) be a reductive dual pair. If \(A \in Q^F_F\) is \(G\)-invariant, let \(\phi_A\) be the
quasi-character of \( \tilde{G} \) attached to \( A \) as in (9.2). For any \( g' \) in \( G' \), we see that \( g'(A) \) will again be \( G \)-invariant, and that \( \phi_A \) and \( \phi_{g'(A)} \) will be equal. Thus if we let \( \mathcal{Q}_k(G, \phi) \) be the set of points \( A \) in \( \mathcal{Q}_k \) which are \( G \)-invariant and such that \( \phi = \phi_A \), then \( \mathcal{Q}_k(G, \phi) \) is a union of \( G' \) orbits.

Since \( \mathcal{Q}(\mathcal{Q}_k(G, \phi)) \) is a compact "subvariety" of \( \mathcal{P}(Y) \), the restriction of the hyperplane section bundle defines a line bundle

\[
L(G, \phi) = L \\
\mathcal{Q}(\mathcal{Q}_k(G, \phi))
\]

Let \( \text{Sec}^\alpha(L) \) denote the space of smooth sections of \( L \). The evaluation of points of the inverse image in \( Y^* \) of \( \mathcal{Q}(\mathcal{Q}_k(G, \phi)) \) defines a linear map \( e: Y \to \text{Sec}^\alpha(L(G, \phi)) \). The evaluation map \( e \) is clearly \( \tilde{G} \) equivariant. We have the dual to \( e \), \( e^*: \text{Sec}^\alpha(L) \to Y^* \). Clearly the image of \( e^* \) will consist of eigenfunctionals for \( \omega_{\gamma}^*(\tilde{G}) \), with eigencharacter \( \phi \).

**Theorem 9.2.** If \( \mathcal{Q}_k(G, \phi) \) is nonempty, then it consists of a single \( G' \) orbit. Further \( L(G, \phi) \) is then a \( G' \) homogeneous line bundle over \( \mathcal{Q}_k(G, \phi) \). Finally, if \( \mu \) is any linear functional on \( Y \) such that \( \omega_{\gamma}^*(\tilde{g})\mu = \phi(\tilde{g})\mu \) for \( \tilde{g} \in \tilde{G} \), then \( \mu = e^*(\nu) \) for some distribution \( \nu \) in \( \text{Sec}^\alpha(L) \).

**10. The spherical case.** \( F \) and other objects are as in \( \S 9 \).

The subgroups of \( H \) comprising \( \mathcal{Q}_k \) are compact modulo the center; because of this, Theorems 9.1 and 9.2 have analogues for \( \mathcal{Q}_k \) which are somewhat more direct. We formulate them.

**Theorem 10.1.** For every \( A \in \mathcal{Q}_k \), there is a vector \( y_A \) in \( Y \), unique up to multiples, such that \( \rho_{\gamma}^*(a)y_A = \chi_A(a)y_A \) for \( a \in A \). The map \( A \to y_A \) is equivariant for the relevant actions of \( \text{Sp} \) on \( \mathcal{Q}_k \) and \( \mathcal{P}(Y) \).

Let \( (G, G') \) be an unramified reductive dual pair in \( \text{Sp} \) and let \( (K, K') \) be a pair of standard maximal compacts of \( K \) and \( K' \), both contained in the standard maximal compact \( J \) of \( \text{Sp} \). As explained in \( \S 7 \), we can consider \( J \), hence \( K \) and \( K' \), to be embedded in \( \tilde{\text{Sp}} \). Let \( L \subseteq W \) be the self-dual lattice fixed by \( J \). Assume that \( \chi \) is unramified in the sense of \( \S 7 \). Then \( A = L \oplus F \subseteq H \) polarizes \( \chi \). Since \( J \) normalizes \( A \), the vector \( y_A \) will be \( \omega_{\gamma}^*(J) \)-invariant up to multiples. Since (except when the residue class field of \( F \) is 3 and \( \dim W = 2 \), a case which can be dealt with separately, but which we will exclude) \( J \) is perfect, we see \( y_A \) will actually be invariant under \( \omega_{\gamma}^* \) by \( J \), and for tortiori by \( K \) and \( K' \).

**Theorem 10.2.** For \( f \in C_{\infty}^\omega(G'/K') \), define \( \gamma(f) = \omega_{\gamma}^*(f)(y_A) \). Then \( \gamma \) is a surjection from \( C_{\infty}^\omega(G'/K') \) to \( I(K) \), the space of \( K \)-fixed vectors in \( Y \).

Since every irreducible admissible representation of \( \tilde{G} \) with a \( K' \) fixed vector is a quotient of \( C_{\infty}^\omega(G'/K') \) in a unique way, and since \( C_{\infty}^\omega(G'/K')^K = C_{\infty}^\omega(G'/K')^K \) is a commutative algebra with 1 (hence a cyclic module for itself) Theorem 7.1 follows from Theorem 9.4.