the finish; so W is now 2 cm from the starting point and 198 cm from the goal. After W crawls for another minute the score is 3 cm traveled and 197 to go; but K stretches, and the distances become 4.5 and 295.5. And so on. Does the worm ever reach the finish? He keeps moving, but the goal seems to move away even faster. (We’re assuming an infinite longevity for K and W, an infinite elasticity of the band, and an infinitely tiny worm.)

Let’s write down some formulas. When K stretches the rubber band, the fraction of it that W has crawled stays the same. Thus he crawls $\frac{1}{100}$th of it the first minute, $\frac{1}{200}$th the second, $\frac{1}{300}$th the third, and so on. After $n$ minutes the fraction of the band that he’s crawled is

$$\frac{1}{100} \left( \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right) = \frac{H_n}{100}. \quad (6.57)$$

So he reaches the finish if $H_n$ ever surpasses 100.

We’ll see how to estimate $H_n$ for large $n$ soon; for now, let’s simply check our analysis by considering how “Superworm” would perform in the same situation. Superworm, unlike W, can crawl 50cm per minute; so she will crawl $\frac{H_n}{2}$ of the band length after $n$ minutes, according to the argument we just gave. If our reasoning is correct, Superworm should finish before $n$ reaches 4, since $H_4 > 2$. And yes, a simple calculation shows that Superworm has only $33 \frac{1}{2}$ cm left to travel after three minutes have elapsed. She finishes in 3 minutes and 40 seconds flat.

Harmonic numbers appear also in Stirling’s triangle. Let’s try to find a closed form for $H_n$, the number of permutations of $n$ objects that have exactly two cycles. Recurrence (6.8) tells us that

$$\left[ \begin{array}{c} n+1 \\ 2 \end{array} \right] = \left[ \begin{array}{c} n \\ 2 \end{array} \right] + \left[ \begin{array}{c} n \\ 1 \end{array} \right]$$

$$= \left[ \begin{array}{c} n \\ 2 \end{array} \right] + (n-1)!, \text{ if } n > 0;$$

and this recurrence is a natural candidate for the summation factor technique of Chapter 2:

$$\frac{1}{n! \left[ \begin{array}{c} n+1 \\ 2 \end{array} \right]} = \frac{1}{(n-1)! \left[ \begin{array}{c} n \\ 2 \end{array} \right] + \frac{1}{n}}.$$

Unfolding this recurrence tells us that $\frac{1}{n! \left[ \begin{array}{c} n+1 \\ 2 \end{array} \right]} = H_n$, hence

$$\left[ \begin{array}{c} n \\ 2 \end{array} \right] = n! H_n \quad (6.58)$$

We proved in Chapter 2 that the harmonic series $\sum_k \frac{1}{k}$ diverges, which means that $H_n$ gets arbitrarily large as $n \to \infty$. But our proof was indirect;