have convergent Taylor series about \( x_0 \), that is, if the functions in Eq. (8) are analytic at \( x = x_0 \). Equations (6) and (7) imply that this will be the case when \( P, Q, \) and \( R \) are polynomials. Any singular point of Eq. (1) that is not a regular singular point is called an irregular singular point of Eq. (1).

In the following sections we discuss how to solve Eq. (1) in the neighborhood of a regular singular point. A discussion of the solutions of differential equations in the neighborhood of irregular singular points is more complicated and may be found in more advanced books.

**Example 5**

In Example 2 we observed that the singular points of the Legendre equation

\[
(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0
\]

are \( x = \pm 1 \). Determine whether these singular points are regular or irregular singular points.

We consider the point \( x = 1 \) first and also observe that on dividing by \( 1 - x^2 \) the coefficients of \( y' \) and \( y \) are \(-2x/(1 - x^2)\) and \( \alpha(\alpha + 1)/(1 - x^2) \), respectively. Thus we calculate

\[
\lim_{x \to 1} \frac{-2x}{1 - x^2} = \lim_{x \to 1} \frac{(x - 1)(-2x)}{(1 - x)(1 + x)} = \lim_{x \to 1} \frac{2x}{1 + x} = 1
\]

and

\[
\lim_{x \to 1} \frac{\alpha(\alpha + 1)}{1 - x^2} = \lim_{x \to 1} \frac{(x - 1)^2 \alpha(\alpha + 1)}{(1 - x)(1 + x)} = \lim_{x \to 1} \frac{(x - 1)(-\alpha)(\alpha + 1)}{1 + x} = 0.
\]

Since these limits are finite, the point \( x = 1 \) is a regular singular point. It can be shown in a similar manner that \( x = -1 \) is also a regular singular point.

**Example 6**

Determine the singular points of the differential equation

\[
2x(x - 2)^2 y'' + 3xy' + (x - 2)y = 0
\]

and classify them as regular or irregular.

Dividing the differential equation by \( 2x(x - 2)^2 \), we have

\[
y'' + \frac{3}{2(x - 2)^2} y' + \frac{1}{2x(x - 2)} y = 0
\]

so \( p(x) = Q(x)/P(x) = 3/(2x - 2)^2 \) and \( q(x) = R(x)/P(x) = 1/(2x(x - 2)) \). The singular points are \( x = 0 \) and \( x = 2 \). Consider \( x = 0 \). We have

\[
\lim_{x \to 0} x p(x) = \lim_{x \to 0} x \frac{3}{2(x - 2)^2} = 0,
\]

\[
\lim_{x \to 0} x^2 q(x) = \lim_{x \to 0} x^2 \frac{1}{2x(x - 2)} = 0.
\]