a little differently, we get a similar upper bound:

\[
\ln n < H_n < \ln n + 1, \quad \text{for } n > 1.
\] (6.60)

We now know the value of \(H_n\) with an error of at most 1.

“Second order” harmonic numbers \(H_n^{(2)}\) arise when we sum the squares of the reciprocals, instead of summing simply the reciprocals:

\[
H_n^{(2)} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} = \sum_{k=1}^{n} \frac{1}{k^2}.
\]

Similarly, we define harmonic numbers of order \(r\) by summing \((-r)\)th powers:

\[
H_n^{(r)} = \sum_{k=1}^{n} \frac{1}{k^r}.
\] (6.61)

If \(r > 1\), these numbers approach a limit as \(n \to \infty\); we noted in Chapter 4 that this limit is conventionally called Riemann’s zeta function:

\[
\zeta(r) = H_n^{(r)} = \sum_{k=1}^{\infty} \frac{1}{k^r}.
\] (6.62)

Euler discovered a neat way to use generalized harmonic numbers to approximate the ordinary ones, \(H_n^{(1)}\). Let’s consider the infinite series

\[
\ln \left( \frac{k}{k-1} \right) = \frac{1}{k} + \frac{1}{2k^2} + \frac{1}{3k^3} + \frac{1}{4k^4} + \cdots,
\] (6.63)

which converges when \(k > 1\). The left-hand side is \(\ln k - \ln (k-1)\); therefore if we sum both sides for \(2 \leq k \leq n\) the left-hand sum telescopes and we get

\[
\ln n - \ln 1 = \sum_{k=2}^{n} \left( \frac{1}{k} + \frac{1}{2k^2} + \frac{1}{3k^3} + \frac{1}{4k^4} + \cdots \right)
= (H_n - 1) + \frac{1}{2}(H_n^{(2)} - 1) + \frac{1}{3}(H_n^{(3)} - 1) + \frac{1}{4}(H_n^{(4)} - 1) + \cdots.
\]