is a second solution of Eq. (1). It is easy to show that \( W(x^{r_1}, x^{r_1} \ln x) = x^{2r_1 - 1} \). Hence \( x^{r_1} \) and \( x^{r_1} \ln x \) are linearly independent for \( x > 0 \), and the general solution of Eq. (1) is

\[
y = (c_1 + c_2 \ln x)x^{r_1}, \quad x > 0.
\]  

(11)

**Example 2**

Solve

\[
x^2 y'' + 5xy' + 4y = 0, \quad x > 0.
\]  

(12)

Substituting \( y = x^r \) in Eq. (12) gives

\[
x^r [r(r - 1) + 5r + 4] = x^r (r^2 + 4r + 4) = 0.
\]

Hence \( r_1 = r_2 = -2 \), and

\[
y = x^{-2}(c_1 + c_2 \ln x), \quad x > 0.
\]  

(13)

**Complex Roots.** Finally, suppose that the roots \( r_1 \) and \( r_2 \) are complex conjugates, say, \( r_1 = \lambda + i\mu \) and \( r_2 = \lambda - i\mu \), with \( \mu \neq 0 \). We must now explain what is meant by \( x^r \) when \( r \) is complex. Remembering that

\[
x^r = e^{r \ln x}
\]  

(14)

when \( x > 0 \) and \( r \) is real, we can use this equation to define \( x^r \) when \( r \) is complex. Then

\[
x^{\lambda + i\mu} = e^{(\lambda + i\mu) \ln x} = e^{\lambda \ln x} e^{i\mu \ln x} = x^\lambda e^{i\mu \ln x} = x^\lambda [\cos(\mu \ln x) + i \sin(\mu \ln x)], \quad x > 0.
\]  

(15)

With this definition of \( x^r \) for complex values of \( r \), it can be verified that the usual laws of algebra and the differential calculus hold, and hence \( x^{r_1} \) and \( x^{r_2} \) are indeed solutions of Eq. (1). The general solution of Eq. (1) is

\[
y = c_1 x^{\lambda + i\mu} + c_2 x^{\lambda - i\mu}.
\]  

(16)

The disadvantage of this expression is that the functions \( x^{\lambda + i\mu} \) and \( x^{\lambda - i\mu} \) are complex-valued. Recall that we had a similar situation for the second order differential equation with constant coefficients when the roots of the characteristic equation were complex. In the same way as we did then we observe that the real and imaginary parts of \( x^{\lambda + i\mu} \), namely,

\[
x^\lambda \cos(\mu \ln x) \quad \text{and} \quad x^\lambda \sin(\mu \ln x),
\]  

(17)

are also solutions of Eq. (1). A straightforward calculation shows that

\[
W[x^\lambda \cos(\mu \ln x), x^\lambda \sin(\mu \ln x)] = \mu x^{2\lambda - 1}.
\]

Hence these solutions are also linearly independent for \( x > 0 \), and the general solution of Eq. (1) is

\[
y = c_1 x^\lambda \cos(\mu \ln x) + c_2 x^\lambda \sin(\mu \ln x), \quad x > 0.
\]  

(18)