nth difference of this polynomial. Almost; first we must clean up a few things. For one, $k^{j-1}$ isn’t a polynomial if $j = 0$; so we will need to split off that term and handle it separately. For another, we’re missing the term $k = 0$ from the formula for nth difference; that term is non-zero when $j = 1$, so we had better restore it (and subtract it out again). The result is

$$U_n = \sum_{j \geq 1} \binom{n}{j} (-1)^{j-1} n^{n-j-1} \sum_{k \geq 0} \binom{n}{k} (-1)^k k^{j-1}$$

$$- \sum_{j \geq 1} \binom{n}{j} (-1)^{j-1} n^{n-j} \binom{n}{0} 0^{j-1}$$

$$- \binom{n}{0} n^j \sum_{k \geq 1} \binom{n}{k} (-1)^k k^{j-1}.$$ 

OK, now the top line (the only remaining double sum) is zero: It’s the sum of multiples of nth differences of polynomials of degree less than $n$, and such nth differences are zero. The second line is zero except when $j = 1$, when it equals $-n^n$. So the third line is the only residual difficulty; we have reduced the original problem to a much simpler sum:

$$U_n = n^n (T_n - 1), \quad \text{where } T_n = \sum \binom{n}{j} (-1)^{j-1}$$  \hspace{1cm} (6.72)$$

For example, $U_3 = \binom{3}{3} \frac{2}{1} = 4 \frac{2}{1}$, $T_3 = \binom{3}{3} \frac{1}{1} = 4 \frac{1}{1} \frac{1}{1} \frac{1}{1} \frac{1}{1} \frac{1}{1} = \frac{1}{0}$; hence $U_3 = 27(T_3 - 1)$ as claimed.

How can we evaluate $T_n$? One way is to replace $\binom{n}{k}$ by $(\frac{n-1}{k}) + (\frac{n-1}{k})$, obtaining a simple recurrence for $T_n$ in terms of $T_{n-1}$. But there’s a more instructive way: We had a similar formula in (5.41), namely

$$\sum_{k} \binom{n}{k} \frac{(-1)^k}{x+k} = \frac{n!}{x(x+1)...(x+n)}.$$ 

If we subtract out the term for $k = 0$ and set $x = 0$, we get $-T_n$. So let’s do it:

$$T_n = \left( \frac{1}{x} - \frac{n!}{x(x+1)...(x+n)} \right) \bigg|_{x=0}$$

$$= \left( \frac{(x+1)...(x+n) - n!}{x(x+1)...(x+n)} \right) \bigg|_{x=0}$$

$$= \left( \frac{x^{n+1}}{n+1} + \ldots + x^{n+1} + \frac{n+1}{1} \cdot n! \right) \bigg|_{x=0} = \frac{1}{n!} \left[ \frac{n+1}{2} \right]$$