where \( \tau_i \) is an irreducible representation of \( \tilde{\Sp}_0 \) of holomorphic type, and \( \sigma_i \) is an irreducible finite-dimensional representation of \( \mathcal{O}_0 \); the representations \( \sigma_i \) and \( \tau_i \) determine each other, and the resulting bijection

\[
\sigma_i \leftrightarrow \tau_i
\]

may be computed using classical invariant theory.

These results, already valid for the pairs

(i) \( (\Sp(2m, R), O(n, R)) \) in \( \Sp(2nm, R) \), with \( n, m \) positive, also extend to cover the pairs

(ii) \( (U_n, U_{p,q}) \) in \( \Sp(2n(p + q), R) \), and the pairs

(iii) \( (\Sp(p, O), O^*(2n)) \) in \( \Sp(4np, R) \).

Thus—in addition to generalizing [Ge 1]—these results also contain earlier works of [GK] (which treats (i) with \( n \geq 2m \), (ii) with \( n \geq 2p \) and \( p = q \), and (iii) with \( p \geq 2n \)) and [KV] (which treats (i) and (ii) with no restrictions).

We shall call the correspondence which results from the “duality” of the pair \( (\Sp, O) \) the duality correspondence.

Now consider an arbitrary local field \( F \) and an arbitrary subgroup \( H \) of \( \tilde{\Sp} \). Let \( R(H) \) denote the set of irreducible smooth representations of \( H \) for which there is a nontrivial \( H \)-intertwining map from the smooth vectors of \( \omega_{\chi} \). Then if \( (G, G') \) is a dual reductive pair in \( \Sp \), \( R(G \cdot G') \) should be the graph of a bijection (or duality correspondence) between \( R(G) \) and \( R(G') \); cf. (3.1) and [Ho 1]. Though unproved in general, this conjecture has been established when \( G \) or \( G' \) is compact. The examples later on give more motivation for the theory.

4. Adelization. Now let \( F \) denote an \( A \)-field not of characteristic 2, \( \nu \) an arbitrary place of \( F \), \( O_\nu \) the ring of integers of \( F_\nu \), \( A \) the adele ring of \( F \), and \( \chi = \prod_\nu \chi_\nu \) a nontrivial character of \( F \setminus A \). If \( V \) is a symplectic space defined over \( F \), then for each \( \nu \) the groups and representations \( H_\nu = H(V/F_\nu) \), \( \Sp_\nu = \Sp(V/F_\nu) \), \( \Sp_\nu \), \( \rho_{\chi_\nu} \), and \( \omega_{\chi_\nu} \) are defined as in \S 2.

As explained in [Ho 1], one can make sense out of the restricted direct product \( \bigotimes \omega_{\chi_\nu} \) and use it to define a multiplier representation \( \omega_{\chi} \) of \( \Sp_A = \Sp(V_A) \). If \( \tilde{\Sp}_A \) denotes the 2-fold cover of \( \Sp_A \) determined by the product of the cocycles defining \( \tilde{\Sp}_0 \), then \( \omega_{\chi} \) also defines an ordinary representation of \( \tilde{\Sp}_A \).

A fundamental property of the global representation \( \omega_{\chi} \) is that it splits over the rational points \( \Sp(F) = \Sp(V/F); \) cf. [We]. If \( (X, Y) \) is a complete polarization of \( V \) and \( \theta \) is the distribution on the Schwartz-Bruhat space \( \mathcal{S}(X(A)) \) defined by \( \theta(\phi) = \sum_{\xi \in X(F)} \phi(\xi) \), then \( \theta \) is \( \Sp(F) \)-invariant. In particular, the functions

\[
\theta_\phi(g) = \theta(\omega_{\chi}(g)\phi)
\]

\[
= \sum_{\xi \in X(F)} (\omega_{\chi}(g)\phi)(\xi)
\]

are automorphic functions on \( \tilde{\Sp}_A \) (slowly increasing continuous functions which are left-\( \Sp(F) \)-invariant).

The basic goal of the global theory of the oscillator representation is to describe the automorphic forms and representations which arise from the \( \theta \)-distribution through functions of the form (4.1). As in the local theory, a great deal of structure is introduced by considering dual reductive pairs in \( \Sp_A \). If \( (G, G') \) is such a pair, it follows from the local theory that there should be a duality correspondence
between representations of $G$ and $G'$ which intertwine with $\omega_\chi$. Moreover, this bijection should pair automorphic representations of $G$ with automorphic representations of $G'$. For a precise description of what to expect in general, see [Ho 1]. Rather than pursue the general theory, we refer the reader to the examples of §6.

5. Local examples. Throughout this section, $F$ is a local field not of characteristic 2, $\chi$ is a fixed character of $F$, $V_0$ is the space $F^2$ equipped with the skew form $x_1y_2 - x_2y_1$, $U_0$ is the orthogonal space $F^*$ with quadratic form $q$, and $V = V_0 \otimes U_0$. Then $\text{Sp}(V_0) = \text{SL}(2, F)$, $O(U_0) = O(q)$, and the problem is to describe $\omega_\chi$ restricted to $\text{SL}(2, F) \cdot O(U_0)$. Recall that we may choose a polarization in $V$ so that $\omega_\chi$ acts in $L^2(F^n)$ according to the formulas

\begin{equation}
\omega_\chi\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} f(X) = \chi(bq(X)) f(X),
\end{equation}

and

\begin{equation}
\omega_\chi\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f(X) = \gamma(q, \chi) f(X).
\end{equation}

The orthogonal group acts linearly in $L^2(F^n)$ through its natural left action in $F^n$.

A. The anisotropic case. Assume that $F$ is nonarchimedean and $q$ is anisotropic. Then $n \leq 4$, $O(q)$ is compact, and

\begin{equation}
\omega_\chi|_{\text{SL}(2, F)} \cdot O(q) = \sum \pi(\sigma) \otimes \sigma.
\end{equation}

Here the sum is over $\sigma$ in $R(O(q))$ and the multiplicity of each $\pi(\sigma) \otimes \sigma$ is one (cf. [RS 1]). It remains to describe $R(O(q))$, $R(\text{SL}(2, F))$, and the resulting duality correspondence.

Case (i): $n = 1$. In this case, $q(x) = ax^2$ with $a \in F^*$, $O(q) = \{ \pm 1 \}$, and the corresponding representations of $\text{SL}(2, F)$ act on the space of even or odd functions in $L^2(F)$. More precisely, if $\sigma$ is the trivial representation of $O(q)$, then $\pi(\sigma)$ (defined by formulas (5.1), (5.2) restricted to the space of even functions) is the unique subrepresentation of an appropriate nonunitary principal series representation of $\text{SL}(2, F)$ at $s = -1/2$; if $\sigma$ is the nontrivial representation of $O(q)$, then $\pi(\sigma)$ (defined by (5.1), (5.2) acting in the space of odd functions) is a supercuspidal representation of $\text{SL}(2, F)$ which is "exceptional" in the sense explained later in §6; for more details, see [Ge 2], [G-PS], and [Ho 3]. Note that $ax^2$ and $bx^2$ lead to equivalent oscillator representations if and only if $ab^{-1} \in (F^*)^2$. In case $F = R$, the pieces of $\omega_\chi$ are square-integrable with extreme vectors of weight $1/2$ and $3/2$.

Case (ii): $n = 2$. Let $U_0$ denote a quadratic extension $K$ of $F$ equipped with the inner product derived from its norm form $q$. Then $O(q)$ is the semidirect product of the norm 1 group $K^1$ in $K$ with the Galois group of $K$ over $F$. In this case each $\sigma$ in $R(O(q))$ can be described in terms of a character of $K^1$ and most of the representations of $R(\text{SL}(2, F))$ are supercuspidal. For further details, see [ST], [Cas], [G], or [RS 1]; in [ST] the correspondence is 2-to-1 because $K^1$ (the special orthogonal group) is used in place of $O(U_0)$. This example is historically important because Shalika and Tanaka were the first to use the oscillator representation to construct an interesting class of irreducible representations.

Case (iii): $n = 3$ [RS 1]. Let $H$ denote the unique division quaternion algebra over $F$, $H^0$ the subspace of pure quaternions, and $q_3$ the restriction of the reduced