The general solution of the Euler equation (1),

\[ x^2y'' + axy' + by = 0, \]

in any interval not containing the origin is determined by the roots \( r_1 \) and \( r_2 \) of the equation

\[ F(r) = r(r-1) + ar + b = 0. \]

If the roots are real and different, then

\[ y = c_1|x|^{r_1} + c_2|x|^{r_2}. \]  \hspace{1cm} (24)

If the roots are real and equal, then

\[ y = (c_1 + c_2 \ln |x|)|x|^{r_1}. \]  \hspace{1cm} (25)

If the roots are complex, then

\[ y = |x|^\lambda [c_1 \cos(\mu \ln |x|) + c_2 \sin(\mu \ln |x|)], \]  \hspace{1cm} (26)

where \( r_1, r_2 = \lambda \pm i\mu. \)

The solutions of an Euler equation of the form

\[ (x - x_0)^2y'' + \alpha(x - x_0)y' + \beta y = 0 \]

are similar to those given in Theorem 5.5.1. If one looks for solutions of the form \( y = (x - x_0)^k \), then the general solution is given by either Eq. (24), (25), or (26) with \( x \) replaced by \( x - x_0 \). Alternatively, we can reduce Eq. (27) to the form of Eq. (1) by making the change of independent variable \( t = x - x_0 \).

The situation for a general second order differential equation with a regular singular point is analogous to that for an Euler equation. We consider that problem in the next section.

**PROBLEMS**

In each of Problems 1 through 12 determine the general solution of the given differential equation that is valid in any interval not including the singular point.

1. \( x^2y'' + 4xy' + 2y = 0 \)
2. \( (x + 1)^2y'' + 3(x + 1)y' + 0.75y = 0 \)
3. \( x^2y' - 3xy' + 4y = 0 \)
4. \( x^2y' + 3xy' + 5y = 0 \)
5. \( x^2y'' - xy' + y = 0 \)
6. \( (x - 1)^2y'' + 8(x - 1)y' + 12y = 0 \)
7. \( x^2y'' + 6xy' - y = 0 \)
8. \( 2x^2y'' - 4xy' + 6y = 0 \)
9. \( x^2y'' - 5xy' + 9y = 0 \)
10. \( (x - 2)^2y'' + 5(x - 2)y' + 8y = 0 \)
11. \( x^2y'' + 2xy' + 4y = 0 \)
12. \( x^2y'' - 4xy' + 4y = 0 \)

In each of Problems 13 through 16 find the solution of the given initial value problem. Plot the graph of the solution and describe how the solution behaves as \( x \to 0 \).

- 13. \( 2x^2y'' + xy' - 3y = 0, \quad y(1) = 1, \quad y'(1) = 4 \)
- 14. \( 4x^2y'' + 8xy' + 17y = 0, \quad y(1) = 2, \quad y'(1) = -3 \)
- 15. \( x^2y'' - 3xy' + 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 3 \)
- 16. \( x^2y'' + 3xy' + 5y = 0, \quad y(1) = 1, \quad y'(1) = -1 \)