The next important sequence of numbers on our agenda is named after Jakob Bernoulli (1654–1705), who discovered curious relationships while working out the formulas for sums of $m$th powers [22]. Let’s write

$$S_m(n) = 0^m + 1^m + \cdots + (n-1)^m = \sum_{k=0}^{n-1} k^m = \sum_{x=0}^{n} x^m \delta x. \quad (6.77)$$

(Thus, when $m > 0$ we have $S_m(n) = H_{n-1}^{(m)}$ in the notation of generalized harmonic numbers.) Bernoulli looked at the following sequence of formulas and spotted a pattern:

$$
\begin{align*}
S_0(n) &= n \\
S_1(n) &= \frac{1}{2}n^2 - \frac{1}{2}n \\
S_2(n) &= \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n \\
S_3(n) &= \frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{1}{4}n^2 \\
S_4(n) &= \frac{1}{5}n^5 - \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n \\
S_5(n) &= \frac{1}{6}n^6 - \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2 \\
S_6(n) &= \frac{1}{7}n^7 - \frac{1}{2}n^6 + \frac{1}{3}n^5 - \frac{1}{8}n^3 + \frac{1}{42}n \\
S_7(n) &= \frac{1}{8}n^8 - \frac{1}{2}n^7 + \frac{7}{24}n^6 - \frac{7}{24}n^4 + \frac{1}{12}n^2 \\
S_8(n) &= \frac{1}{9}n^9 - \frac{1}{2}n^8 + \frac{2}{3}n^7 - \frac{7}{18}n^5 + \frac{7}{9}n^3 - \frac{1}{30}n \\
S_9(n) &= \frac{1}{10}n^{10} - \frac{1}{2}n^9 + \frac{3}{4}n^8 - \frac{7}{10}n^6 + \frac{1}{4}n^4 - \frac{3}{20}n^2 \\
S_{10}(n) &= \frac{1}{11}n^{11} - \frac{1}{2}n^{10} + \frac{5}{6}n^9 - n^7 + \frac{1}{3}n^5 - \frac{1}{11}n^3 + \frac{5}{66}n
\end{align*}
$$

Can you see it too? The coefficient of $n^m+1$ in $S_m(n)$ is always $1/(m+1)$. The coefficient of $n^m$ is always $-1/2$. The coefficient of $n^{m-1}$ is always $\ldots$ let’s see $\ldots$ $m/12$. The coefficient of $n^{m-2}$ is always zero. The coefficient of $n^{m-3}$ is always $\ldots$ let’s see $\ldots$ hmmm $\ldots$ yes, it’s $-m(m-1)(m-2)/720$. The coefficient of $n^{m-4}$ is always zero. And it looks as if the pattern will continue, with the coefficient of $n^{m-k}$ always being some constant times $m^k$.

That was Bernoulli’s discovery. In modern notation we write the coefficients in the form

$$S_m(n) = \frac{1}{m+1} \left( B_0 n^m + \binom{m+1}{1} B_1 n^{m-1} + \cdots + \binom{m+1}{m} B_m n \right) = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k n^{m+1-k}. \quad (6.78)$$