ating the network can be expressed through a particular parametric form $P(G| \theta)$, where $G$ is the observed graph and $\theta$ is a parameter vector. For example, typical formulations of exponential random graph models assume that the building blocks of real networks are given by such structures as $k$-stars and $k$-triangles, with different weights assigned to different structures, whereas Kronecker graphs assume that the number of edges and the number of nodes in a given network are related by a densification power law with a suitable densification parameter. In such frameworks, estimating the model from data reduces to fitting the model parameters, whereas the model structure remains fixed from the very beginning. The problem is that, in order for a parametric model to deliver an accurate estimate of the distribution at hand, its prior assumption concerning the form of the modeled distribution must be satisfied by the given data, which is something that we can rarely assess a priori. To date, the knowledge we have concerning real-world network phenomena does not allow to assume that any particular parametric assumption is really capturing in depth any network-generating law, although some observed properties may happen to be modeled fairly well.

The aim of this paper is twofold. On the one hand, we take a first step toward nonparametric modeling of random networks by developing a novel network statistic, which we call the Fiedler delta statistic. The Fiedler delta function allows to model different graph properties at once in an extremely compact form. This statistic is based on the spectral analysis of the graph Laplacian. In particular, it is based on the smallest non-zero eigenvalue of the Laplacian matrix, which is known as Fiedler value [Fiedler, 1973, Mohar, 1991].

On the other hand, we systematically adopt a conditional approach to random graph modeling, i.e. we focus on the conditional distribution of edges given some neighboring portion of the network, while setting aside the problem of estimating joint distributions. The resulting conditional random graph model is what we call Fiedler random graph (FRG). Roughly speaking, to model the conditional distribution of an edge variable $X_{uv}$ with respect to its neighborhood, we compute the difference in Fiedler values for the vicinity subgraph including or excluding the edge $\{u,v\}$. The underlying intuition is that the variations encapsulated in the Fiedler delta for each particular edge will give a measure of the role of that edge in determining the algebraic connectivity of its neighborhood. As part of our contributions, we theoretically prove that FRGs capture edge dependencies at any distance within a given neighborhood, hence defining a fairly general class of conditional probability distributions over graphs.

Experiments on the estimation of (conditional) link distributions in large-scale networks show that FRGs are well suited for estimation problems on very large networks, especially small-world and scale-free networks. Our results reveal that the FRG model supersedes other approaches in terms of edge prediction accuracy, while allowing for efficient computation in the large-scale setting. In particular, it is known that the computation of the Fiedler delta scales polynomially in the size of the analyzed neighborhood [Bai et al., 2000]. And the experiments show that even for small sizes of neighborhoods (which allow for extremely fast computation), our model regularly outperforms the alternative ones—which we reformulate here in terms of conditional models so as to allow for transparent comparison.

The paper is organized as follows. Sec. 2 reviews some preliminary notions concerning the Laplacian spectrum of graphs. The FRG model is presented in Sec. 3, where we also show how to estimate FRGs from data, and we analyze the dependence structures involved in FRGs. In Sec. 4 we review (and to some extent develop) a few alternative conditional approaches to random graph estimation, starting from some well-known generative models. All considered models are then evaluated experimentally in Sec. 5, while Sec. 6 draws some conclusions and sketches a few directions for further work.

## 2 Graphs, Laplacians, and Eigenvalues

Let $G = (V, E)$ be an undirected graph with $n$ nodes. In the following we assume that the graph is unweighted with adjacency matrix $A$. For each (unordered) pair of nodes $\{u, v\}$ such that $u \neq v$, we take $X_{uv}$ to denote a binary random variable such that $X_{uv} = 1$ if $\{u, v\} \in E$, and $X_{uv} = 0$ otherwise. Since the graph is undirected, $X_{uv} = X_{vu}$.

The degree $d_u$ of a node $u \in V$ is defined as the number of connections of $u$ to other nodes, that is $d_u = |\{v: \{u, v\} \in E\}|$. Accordingly, the degree matrix $D$ of a graph $G$ corresponds to the diagonal matrix with the vertex degrees $d_1, \ldots, d_n$ on the diagonal. The main tools exploited by the random graph model proposed here are the graph Laplacian matrices. Different graph Laplacians have been identified in the literature [von Luxburg, 2007]. In this paper, we use consistently the unnormalized graph Laplacian, defined as $L = D - A$. The basic facts related to the unnormalized Laplacian matrix can be summarized as follows:

**Proposition 1** (Mohar [1991]). The unnormalized graph Laplacian $L$ of an undirected graph $G$ with $n$