Recurrence (6.95) gives us a simple way to calculate Bernoulli numbers, via tangent numbers, using only simple operations on integers; by contrast, the defining recurrence (6.79) involves difficult arithmetic with fractions.

If we want to compute the sum of nth powers from a to b - 1 instead of from 0 to n - 1, the theory of Chapter 2 tells us that

\[ \sum_{k=a}^{b-1} k^m = \sum_{x=a}^{b} x^m \delta x = S_m(b) - S_m(a). \]  

(6.96)

This identity has interesting consequences when we consider negative values of k: We have

\[ \sum_{k=-n+1}^{-1} k^m = (-1)^m \sum_{k=0}^{n-1} k^m, \quad \text{when } m > 0, \]

hence

\[ S_m(0) - S_m(-n + 1) = (-1)^m (S_m(n) - S_m(O)). \]

But \( S_m(0) = 0 \), so we have the identity

\[ S_m(1 - n) = (-1)^{m+1} S_m(n), \quad m > 0. \]  

(6.97)

Therefore \( S_m(1) = 0 \). If we write the polynomial \( S_m(n) \) in factored form, it will always have the factors \( n \) and \( (n - 1) \), because it has the roots 0 and 1. In general, \( S_m(n) \) is a polynomial of degree \( m + 1 \) with leading term \( \frac{1}{m+1} n^{m+1} \). Moreover, we can set \( n = \frac{1}{2} \) in (6.97) to get \( S_m(\frac{1}{2}) = (-1)^{m+1} S_m(\frac{1}{2}) \); if \( m \) is even, this makes \( S_m(\frac{1}{2}) = 0 \), so \( n = \frac{1}{2} \) will be an additional factor. These observations explain why we found the simple factorization

\[ S_2(n) = \frac{1}{3} n(n - \frac{1}{2})(n - 1) \]

in Chapter 2; we could have used such reasoning to deduce the value of \( S_2(n) \) without calculating it! Furthermore, (6.97) implies that the polynomial with the remaining factors, \( S_m(n) = S_m(n)/(n - \frac{1}{2}) \), always satisfies

\[ S_m(1 - n) = S_m(n), \quad m \text{ even}, \quad m > 0. \]

It follows that \( S_m(n) \) can always be written in the factored form

\[ S_m(n) = \begin{cases} \frac{1}{m+1} \prod_{k=1}^{\lfloor m/2 \rfloor} (n - \frac{1}{2} - \alpha_k)(n - \frac{1}{2} + \alpha_k), & m \text{ odd}; \\ \frac{(n - \frac{1}{2})^{m/2}}{m+1} \prod_{k=1}^{\lfloor m/2 \rfloor} (n - \frac{1}{2} - \alpha_k)(n - \frac{1}{2} + \alpha_k), & m \text{ even}. \end{cases} \]  

(6.98)