integer, then the second solution normally has a more complicated structure. In all cases, though, it is possible to find at least one solution of the form (7) or (22). If \( r_1 \) and \( r_2 \) differ by an integer, this solution corresponds to the larger value of \( r \). If there is only one such solution, then the second solution involves a logarithmic term, just as for the Euler equation when the roots of the characteristic equation are equal. The method of reduction of order or some other procedure can be invoked to determine the second solution in such cases. This is discussed in Sections 5.7 and 5.8.

If the roots of the indicial equation are complex, then they cannot be equal or differ by an integer, so there are always two solutions of the form (7) or (22). Of course, these solutions are complex-valued functions of \( x \). However, as for the Euler equation, it is possible to obtain real-valued solutions by taking the real and imaginary parts of the complex solutions.

Finally, we mention a practical point. If \( P, Q, \) and \( R \) are polynomials, it is often much better to work directly with Eq. (1) than with Eq. (3). This avoids the necessity of expressing \( x^\alpha Q(x)/P(x) \) and \( x^\beta R(x)/P(x) \) as power series. For example, it is more convenient to consider the equation

\[
x(1 + x)y'' + 2y' + xy = 0
\]

than to write it in the form

\[
x^2y'' + \frac{2x}{1 + x}y' + \frac{x^2}{1 + x}y = 0,
\]

which would entail expanding \( 2x/(1 + x) \) and \( x^2/(1 + x) \) in power series.

## PROBLEMS

In each of Problems 1 through 10 show that the given differential equation has a regular singular point at \( x = 0 \). Determine the indicial equation, the recurrence relation, and the roots of the indicial equation. Find the series solution \( x > 0 \) corresponding to the larger root. If the roots are unequal and do not differ by an integer, find the series solution corresponding to the smaller root also.

1. \( 2xy'' + y' + xy = 0 \)
2. \( x^2y'' + xy' + (x^2 - \frac{1}{5})y = 0 \)
3. \( xy'' + y = 0 \)
4. \( xy'' + y' - y = 0 \)
5. \( 3x^2y'' + 2xy' + x^2y = 0 \)
6. \( x^2y'' + xy' + (x - 2)y = 0 \)
7. \( xy'' + (1 - x)y' - y = 0 \)
8. \( 2x^2y'' + 3xy' + (2x^2 - 1)y = 0 \)
9. \( x^2y'' - x(x + 3)y' + (x + 3)y = 0 \)
10. \( x^2y'' + (x^2 + \frac{1}{2})y = 0 \)
11. The Legendre equation of order \( \alpha \) is

\[
(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0.
\]

The solution of this equation near the ordinary point \( x = 0 \) was discussed in Problems 22 and 23 of Section 5.3. In Example 5 of Section 5.4 it was shown that \( x = \pm 1 \) are regular singular points. Determine the indicial equation and its roots for the point \( x = 1 \). Find a series solution in powers of \( x - 1 \) for \( x - 1 > 0 \).

**Hint:** Write \( 1 + x = 2 + (x - 1) \) and \( x = 1 + (x - 1) \). Alternatively, make the change of variable \( x - 1 = t \) and determine a series solution in powers of \( t \).

12. The Chebyshev equation is

\[
(1 - x^2)y'' - xy' + \alpha^2y = 0,
\]

where \( \alpha \) is a constant; see Problem 10 of Section 5.3.

(a) Show that \( x = 1 \) and \( x = -1 \) are regular singular points, and find the exponents at each of these singularities.

(b) Find two linearly independent solutions about \( x = 1 \).