Here \( \alpha_1 = \frac{1}{2} \), and \( \alpha_2, \ldots, \alpha_{[m/2]} \) are appropriate complex numbers whose values depend on \( m \). For example,

\[
\begin{align*}
S_3(n) &= n^2(n-1)^2/4; \\
S_4(n) &= n(n-\frac{1}{2})(n-1)(n-\frac{1}{2} + \sqrt{3}/2)(n-\frac{1}{2} - \sqrt{3}/2)/5; \\
S_5(n) &= n^2(n-1)^2(n-\frac{1}{2} + \sqrt{3}/2)(n-\frac{1}{2} - \sqrt{3}/2)/6; \\
S_6(n) &= n(n-\frac{1}{2})(n-1)(n-\frac{1}{2} + \alpha)(n-\frac{1}{2} - \alpha)(n-\frac{1}{2} + \overline{\alpha})(n-\frac{1}{2} - \overline{\alpha}),
\end{align*}
\]

where \( \alpha = 2^{-5/2}3^{-1/2}31^{1/4}(\sqrt{3} + \sqrt{27} + i \sqrt{3} - \sqrt{27}) \).

If \( m \) is odd and greater than 1, we have \( B_m = 0 \); hence \( S_m(n) \) is divisible by \( n^2 \) (and by \( (n-1)^2 \)). Otherwise the roots of \( S_m(n) \) don’t seem to obey a law.

Let’s conclude our study of Bernoulli numbers by looking at how they relate to Stirling numbers. One way to compute \( S_m(n) \) is to change ordinary powers to falling powers, since the falling powers have easy sums. After doing those easy sums we can convert back to ordinary powers:

\[
S_m(n) = \sum_{k=0}^{n-1} k^m = \sum_{k=0}^{n-1} \sum_{j \geq 0} \binom{m}{j} k^j = \sum_{j \geq 0} \binom{m}{j} \sum_{k=0}^{n-1} k^j
\]

Therefore, equating coefficients with those in (6.78), we must have the identity

\[
\sum_{j \geq 0} \binom{m}{j} \left[ \frac{1}{j+1} \right] \frac{(-1)^{j+1-k}}{j+1} = \frac{1}{m+1} \binom{m+1}{k} B_{m+1-k}.
\]  

(6.99)

It would be nice to prove this relation directly, thereby discovering Bernoulli numbers in a new way. But the identities in Tables 250 or 251 don’t give us any obvious handle on a proof by induction that the left-hand sum in (6.99) is a constant times \( m^{k-1} \). If \( k = m + 1 \), the left-hand sum is just \( \binom{m}{m+1}/(m+1) = 1/(m+1) \), so that case is easy. And if \( k = m \), the left-hand side sums to \( \frac{m}{m-1} \frac{m}{m-2} \cdots \frac{m}{m-(m-1)}(m+1)\), which is \( \frac{1}{4}(m-1) - \frac{1}{2} m = \frac{1}{2} \); so that case is pretty easy too. But if \( k < m \), the left-hand sum looks hairy. Bernoulli would probably not have discovered his numbers if he had taken this route.