where
\[ xp(x) = \sum_{n=0}^{\infty} p_n x^n, \quad x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n, \] (2)
and both series converge in an interval \(|x| < \rho\) for some \(\rho > 0\). The point \(x = 0\) is a regular singular point, and the corresponding Euler equation is
\[ x^2 y'' + p_0 x y' + q_0 y = 0. \] (3)
We seek a solution of Eq. (1) for \(x > 0\) and assume that it has the form
\[ y = \phi(r, x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{r+n}, \] (4)
where \(a_0 \neq 0\), and we have written \(y = \phi(r, x)\) to emphasize that \(\phi\) depends on \(r\) as well as \(x\). It follows that
\[ y' = \sum_{n=0}^{\infty} (r+n) a_n x^{r+n-1}, \quad y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n x^{r+n-2}. \] (5)
Then, substituting from Eqs. (2), (4), and (5) in Eq. (1) gives
\begin{align*}
a_0 F(r) x^r &+ a_1 (r+1) x^{r+1} + \cdots + a_n (r+n)(r+n-1) x^{r+n} + \cdots \\
&+ (p_0 + p_1 x + \cdots + p_n x^n + \cdots) \\
&\times [a_0 r x^r + a_1 (r+1) x^{r+1} + \cdots + a_n (r+n) x^{r+n} + \cdots] \\
&+ (q_0 + q_1 x + \cdots + q_n x^n + \cdots) \\
&\times (a_0 x^r + a_1 x^{r+1} + \cdots + a_n x^{r+n} + \cdots) = 0.
\end{align*}
Multiplying the infinite series together and then collecting terms, we obtain
\begin{align*}
a_0 F(r) x^r &+ [a_1 F(r+1) + a_0 (p_1 r + q_1)] x^{r+1} \\
&+ [a_2 F(r+2) + a_0 (p_2 r^2 + q_2) + a_1 (p_1 (r+1) + q_1)] x^{r+2} \\
&+ \cdots + [a_n F(r+n) + a_0 (p_n r^n + q_n) + a_1 (p_{n-1} (r+1) + q_{n-1})] \\
&+ \cdots + a_{n-1} [p_1 (r+n-1) + q_1] x^{r+n} + \cdots = 0,
\end{align*}
or in a more compact form,
\[ L[\phi](r, x) = a_0 F(r) x^r \\
+ \sum_{n=1}^{\infty} \left\{ F(r+n) a_n + \sum_{k=0}^{n-1} a_k [(r+k) p_{n-k} + q_{n-k}] \right\} x^{r+n} = 0, \] (6)
where
\[ F(r) = r(r-1) + p_0 r + q_0. \] (7)
For Eq. (6) to be satisfied identically the coefficient of each power of \(x\) must be zero.
Since \(a_0 \neq 0\), the term involving \(x^r\) yields the equation \(F(r) = 0\). This equation is called the indicial equation. Note that it is exactly the equation we would obtain in looking for solutions \(y = x^r\) of the Euler equation (3). Let us denote the roots of the indicial equation by \(r_1\) and \(r_2\) with \(r_1 \geq r_2\) if the roots are real. If the roots are complex, the designation of the roots is immaterial. Only for these values of \(r\) can we expect to find solutions of Eq. (1) of the form (4). The roots \(r_1\) and \(r_2\) are called the