cases are:

\[ a_0 = 1 \quad a_1 = 2 \quad a_2 = 3 \quad a_3 = 5 \]

When \( n \) is even, we have an even number of bounces and the ray passes through; when \( n \) is odd, the ray is reflected and it re-emerges on the same side it entered. The \( a_n \)'s seem to be Fibonacci numbers, and a little staring at the figure tells us why: For \( n \geq 2 \), the \( n \)-bounce rays either take their first bounce off the opposite surface and continue in \( a_{n-1} \) ways, or they begin by bouncing off the middle surface and then bouncing back again to finish in \( a_{n-2} \) ways. Thus we have the Fibonacci recurrence \( a_n = a_{n-1} + a_{n-2} \).

The initial conditions are different, but not very different, because we have \( a_0 = 1 = F_2 \) and \( a_1 = 2 = F_3 \); therefore everything is simply shifted two places, and \( a_n = F_{n+2} \).

Leonardo Fibonacci introduced these numbers in 1202, and mathematicians gradually began to discover more and more interesting things about them. Édouard Lucas, the perpetrator of the Tower of Hanoi puzzle discussed in Chapter 1, worked with them extensively in the last half of the nineteenth century (in fact it was Lucas who popularized the name “Fibonacci numbers”). One of his amazing results was to use properties of Fibonacci numbers to prove that the 39-digit Mersenne number \( 2^{127} - 1 \) is prime.

One of the oldest theorems about Fibonacci numbers, due to the French astronomer Jean-Dominique Cassini in 1680 [45], is the identity

\[ F_{n+1} F_{n-1} - F_n^2 = (-1)^n, \quad \text{for } n > 0. \tag{6.103} \]

When \( n = 6 \), for example, Cassini’s identity correctly claims that \( 1 \cdot 3 \cdot 5 - 8^2 = 1 \).

A polynomial formula that involves Fibonacci numbers of the form \( F_{n+k} \) for small values of \( k \) can be transformed into a formula that involves only \( F_n \) and \( F_{n+1} \), because we can use the rule

\[ F_m = F_{m+2} - F_{m+1} \tag{6.104} \]

to express \( F_m \) in terms of higher Fibonacci numbers when \( m < n \), and we can use

\[ F_m = F_{m-2} + F_{m-1} \tag{6.105} \]

to replace \( F_m \) by lower Fibonacci numbers when \( m > n+1 \). Thus, for example, we can replace \( F_{n-1} \) by \( F_{n+1} - F_n \) in (6.103) to get Cassini’s identity in the