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form

\[ F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n. \]  

(6.106)

Moreover, Cassini’s identity reads

\[ F_{n+2}F_n - F_{n+1}^2 = (-1)^{n+1} \]

when \( n \) is replaced by \( n + 1 \); this is the same as \( (F_{n+1} + F_n)F_n - F_{n+1}^2 = (-1)^{n+1} \), which is the same as (6.106). Thus Cassini(n) is true if and only if Cassini(n+1) is true; equation (6.103) holds for all \( n \) by induction.

Cassini’s identity is the basis of a geometrical paradox that was one of Lewis Carroll’s favorite puzzles [54], [258], [298]. The idea is to take a chess-board and cut it into four pieces as shown here, then to reassemble the pieces into a rectangle:

![Chessboard diagram]

Presto: The original area of \( 8 \times 8 = 64 \) squares has been rearranged to yield \( 5 \times 13 = 65 \) squares! A similar construction dissects any \( F_n \times F_n \) square into four pieces, using \( F_{n+1}, F_n, F_{n+1}, \) and \( F_{n+2} \) as dimensions wherever the illustration has 13, 8, 5, and 3 respectively. The result is an \( F_{n+1} \times F_{n+2} \) rectangle; by (6.103), one square has therefore been gained or lost, depending on whether \( n \) is even or odd.

Strictly speaking, we can’t apply the reduction (6.105) unless \( m \geq 2 \), because we haven’t defined \( F_n \) for negative \( n \). A lot of maneuvering becomes easier if we eliminate this boundary condition and use (6.104) and (6.105) to define Fibonacci numbers with negative indices. For example, \( F_1 \) turns out to be \( F_1 = 1; \) then \( F_2 \) is \( F_0 - F_1 = -1. \) In this way we deduce the values

\[
\begin{array}{cccccccccccc}
-11 & -10 & -9 & -8 & -7 & -6 & -5 & -4 & -3 & -2 & -1 & 0 \\
F_n & 89 & 55 & 34 & 21 & 13 & 8 & 5 & 3 & 2 & 1 & 0
\end{array}
\]

and it quickly becomes clear (by induction) that

\[ F_{-n} = (-1)^{n-1}F_n, \quad \text{integer } n. \]  

(6.107)

Cassini’s identity (6.103) is true for all integers \( n \), not just for \( n > 0 \), when we extend the Fibonacci sequence in this way.