whose coefficients are given by
\[ p_0 = \lim_{x \to 0} xp(x), \quad q_0 = \lim_{x \to 0} x^3 q(x). \tag{12} \]

Note that these are exactly the limits that must be evaluated in order to classify the singularity as a regular singular point; thus they have usually been determined at an earlier stage of the investigation.

Further, if \( x = 0 \) is a regular singular point of the equation
\[ P(x)y'' + Q(x)y' + R(x)y = 0, \tag{13} \]
where the functions \( P, Q, \) and \( R \) are polynomials, then \( xp(x) = xQ(x)/P(x) \) and \( x^2q(x) = x^2R(x)/P(x) \). Thus
\[ p_0 = \lim_{x \to 0} x \frac{Q(x)}{P(x)}, \quad q_0 = \lim_{x \to 0} x^2 \frac{R(x)}{P(x)}. \tag{14} \]

Finally, the radii of convergence for the series in Eqs. (9) and (10) are at least equal to the distance from the origin to the nearest zero of \( P \) other than \( x = 0 \) itself.

**Example 1**

Discuss the nature of the solutions of the equation
\[ 2x(1 + x)y'' + (3 + x)y' - xy = 0 \]
near the singular points.

This equation is of the form (13) with \( P(x) = 2x(1 + x), Q(x) = 3 + x, \) and \( R(x) = -x \). The points \( x = 0 \) and \( x = -1 \) are the only singular points. The point \( x = 0 \) is a regular singular point, since
\[ \lim_{x \to 0} x Q(x) = \lim_{x \to 0} x \frac{3!}{2x(1 + x)} = \frac{3}{2}, \]
\[ \lim_{x \to 0} x^2 R(x) = \lim_{x \to 0} x^2 \frac{-x}{2x(1 + x)} = 0. \]

Further, from Eq. (14), \( p_0 = \frac{3}{2} \) and \( q_0 = 0 \). Thus the indicial equation is \( r(r - 1) + \frac{3}{2} = 0 \), and the roots are \( r_1 = 0, r_2 = -\frac{1}{2} \). Since these roots are not equal and do not differ by an integer, there are two linearly independent solutions of the form
\[ y_1(x) = 1 + \sum_{n=1}^{\infty} a_n (0)x^n \quad \text{and} \quad y_2(x) = |x|^{-1/2} \left[ 1 + \sum_{n=1}^{\infty} a_n (-\frac{1}{2})x^n \right] \]
for \( 0 < |x| < \rho \). A lower bound for the radius of convergence of each series is 1, the distance from \( x = 0 \) to \( x = -1 \), the other zero of \( P(x) \). Note that the solution \( y_1 \) is bounded as \( x \to 0 \), indeed is analytic there, and that the second solution \( y_2 \) is unbounded as \( x \to 0 \).

The point \( x = -1 \) is also a regular singular point, since
\[ \lim_{x \to -1} (x + 1) \frac{Q(x)}{P(x)} = \lim_{x \to -1} \frac{(x + 1)(3 + x)}{2x(1 + x)} = -1, \]
\[ \lim_{x \to -1} (x + 1)^2 \frac{R(x)}{P(x)} = \lim_{x \to -1} \frac{(x + 1)^2(-x)}{2x(1 + x)} = 0. \]