In this case \( p_0 = -1, q_0 = 0 \), so the indicial equation is \( r(r - 1) - r = 0 \). The roots of the indicial equation are \( r_1 = 2 \) and \( r_2 = 0 \). Corresponding to the larger root there is a solution of the form

\[
y_1(x) = (x + 1)^2 \left[ 1 + \sum_{n=1}^{\infty} a_n(2)(x + 1)^n \right].
\]

The series converges at least for \( |x + 1| < 1 \) and \( y_1 \) is an analytic function there. Since the two roots differ by a positive integer, there may or may not be a second solution of the form

\[
y_2(x) = 1 + \sum_{n=1}^{\infty} a_n(0)(x + 1)^n.
\]

We cannot say more without further analysis.

Observe that no complicated calculations were required to discover the information about the solutions presented in this example. All that was needed was to evaluate a few limits and solve two quadratic equations.

We now consider the cases in which the roots of the indicial equation are equal, or differ by a positive integer, \( r_1 - r_2 = N \). As we have shown earlier, there is always one solution of the form \( (9) \) corresponding to the larger root \( r_1 \) of the indicial equation. By analogy with the Euler equation, we might expect that if \( r_1 = r_2 \), then the second solution contains a logarithmic term. This may also be true if the roots differ by an integer.

**Equal Roots.** The method of finding the second solution is essentially the same as the one we used in finding the second solution of the Euler equation (see Section 5.5) when the roots of the indicial equation were equal. We consider \( r \) to be a continuous variable and determine \( a_n \) as a function of \( r \) by solving the recurrence relation \( (8) \). For this choice of \( a_n(r) \) for \( n \geq 1 \), Eq. \( (6) \) reduces to

\[
L[\phi](r, x) = a_0 F(r) x^{r'} = a_0 (r - r_1)^2 x^{r'},
\]

since \( r_1 \) is a repeated root of \( F(r) \). Setting \( r = r_1 \) in Eq. \( (15) \), we find that \( L[\phi](r_1, x) = 0 \); hence, as we already know, \( y_1(x) \) given by Eq. \( (9) \) is one solution of Eq. \( (1) \). But more important, it also follows from Eq. \( (15) \), just as for the Euler equation, that

\[
L \left[ \frac{\partial \phi}{\partial r} \right] (r_1, x) = a_0 \frac{\partial}{\partial r} \left[ x^{r'} (r - r_1)^2 \right] \bigg|_{r=r_1} = a_0 (r - r_1)^2 x^{r'} \ln x + 2(r - r_1)x^{r'} \bigg|_{r=r_1} = 0.
\]

Hence, a second solution of Eq. \( (1) \) is

\[
y_2(x) = \frac{\partial \phi(r, x)}{\partial r} \bigg|_{r=r_1} = \frac{\partial}{\partial r} \left\{ x^{r'} \left[ a_0 + \sum_{n=1}^{\infty} a_n(r) x^n \right] \right\} \bigg|_{r=r_1} = (x^{r_1} \ln x) \left[ a_0 + \sum_{n=1}^{\infty} a_n(r_1) x^n \right] + x^{r_1} \sum_{n=1}^{\infty} a'_n(r_1) x^n = y_1(x) \ln x + x^{r_1} \sum_{n=1}^{\infty} a'_n(r_1) x^n, \quad x > 0,
\]

(17)