The process of reducing $F_{n+k}$ to a combination of $F_n$ and $F_{n+1}$ by using (6.105) and (6.104) leads to the sequence of formulas

\[
\begin{align*}
F_{n+2} &= F_{n+1} + F_n & F_{n-1} &= F_{n+1} - F_n \\
F_{n+3} &= 2F_{n+1} + F_n & F_{n-2} &= -F_{n+1} + 2F_n \\
F_{n+4} &= 3F_{n+1} + 2F_n & F_{n-3} &= 2F_{n+1} - 3F_n \\
F_{n+5} &= 5F_{n+1} + 3F_n & F_{n-4} &= -3F_{n+1} + 5F_n
\end{align*}
\]

in which another pattern becomes obvious:

\[
F_{n+k} = F_k F_{n+1} + F_{k-1} F_n.
\]

This identity, easily proved by induction, holds for all integers $k$ and $n$ (positive, negative, or zero).

If we set $k = n$ in (6.108), we find that

\[
F_{2n} = F_n F_{n+1} + F_{n-1} F_n;
\]

hence $F_{2n}$ is a multiple of $F_n$. Similarly,

\[
F_{3n} = F_{2n} F_{n+1} + F_{2n-1} F_n,
\]

and we may conclude that $F_{3n}$ is also a multiple of $F_n$. By induction,

\[
F_{kn} \text{ is a multiple of } F_n,
\]

for all integers $k$ and $n$. This explains, for example, why $F_{15}$ (which equals 610) is a multiple of both $F_3$ and $F_5$ (which are equal to 2 and 5). Even more is true, in fact; exercise 27 proves that

\[
\gcd(F_m, F_n) = F_{\gcd(m,n)},
\]

For example, $\gcd(F_{12}, F_{18}) = \gcd(144, 2584) = 8 = F_6$.

We can now prove a converse of (6.110): If $n > 2$ and if $F_m$ is a multiple of $F_n$, then $m$ is a multiple of $n$. For if $F_m \mid F_n$ then $F_n \mid \gcd(F_m, F_n) = F_{\gcd(m,n)} \leq F_n$. This is possible only if $F_{\gcd(m,n)} = F_n$; and our assumption that $n > 2$ makes it mandatory that $\gcd(m, n) = n$. Hence $n \mid m$.

An extension of these divisibility ideas was used by Yuri Matijasevich in his famous proof [213] that there is no algorithm to decide if a given multivariate polynomial equation with integer coefficients has a solution in integers. Matijasevich’s lemma states that, if $n > 2$, the Fibonacci number $F_m$ is a multiple of $F_n^2$ if and only if $m$ is a multiple of $n F_n$.

Let’s prove this by looking at the sequence $(F_{kn} \mod F_n^2)$ for $k = 1, 2, 3, \ldots$, and seeing when $F_{kn} \mod F_n^2 = 0$. (We know that $m$ must have the