where $a'_n(r_1)$ denotes $da_n/dr$ evaluated at $r = r_1$.

It may turn out that it is difficult to determine $a_n(r)$ as a function of $r$ from the recurrence relation (8) and then to differentiate the resulting expression with respect to $r$. An alternative is simply to assume that $y$ has the form of Eq. (17), that is,

$$y = y_1(x) \ln x + x^{r_1} \sum_{n=1}^{\infty} b_n x^n, \quad x > 0,$$

where $y_1(x)$ has already been found. The coefficients $b_n$ are calculated, as usual, by substituting into the differential equation, collecting terms, and setting the coefficient of each power of $x$ equal to zero. A third possibility is to use the method of reduction of order to find $y_2(x)$ once $y_1(x)$ is known.

**Roots Differing by an Integer.** For this case the derivation of the second solution is considerably more complicated and will not be given here. The form of this solution is stated in Eq. (24) in the following theorem. The coefficients $c_n(r_2)$ in Eq. (24) are given by

$$c_n(r_2) = \left. \frac{d}{dr} [(r - r_2)a_n(r)] \right|_{r=r_2}, \quad n = 1, 2, \ldots,$$

where $a_n(r)$ is determined from the recurrence relation (8) with $a_0 = 1$. Further, the coefficient $a$ in Eq. (24) is

$$a = \lim_{r \to r_2} (r - r_2)a_N(r).$$

If $a_N(r_2)$ is finite, then $a = 0$ and there is no logarithmic term in $y_2$. A full derivation of the formulas (19), (20) may be found in Coddington Chapter 4.

In practice the best way to determine whether $a$ is zero in the second solution is simply to try to compute the $a_n$ corresponding to the root $r_2$ and to see whether it is possible to determine $a_N(r_2)$. If so, there is no further problem. If not, we must use the form (24) with $a \neq 0$.

When $r_1 - r_2 = N$, there are again three ways to find a second solution. First, we can calculate $a$ and $c_n(r_2)$ directly by substituting the expression (24) for $y$ in Eq. (1). Second, we can calculate $c_n(r_2)$ and $a$ of Eq. (24) using the formulas (19) and (20). If this is the planned procedure, then in calculating the solution corresponding to $r = r_1$ be sure to obtain the general formula for $a_n(r)$ rather than just $a_n(r_1)$. The third alternative is to use the method of reduction of order.

**Theorem 5.7.1** Consider the differential equation (1),

$$x^2 y'' + x[xp(x)]y' + [x^2 q(x)]y = 0,$$

where $x = 0$ is a regular singular point. Then $xp(x)$ and $x^2 q(x)$ are analytic at $x = 0$ with convergent power series expansions

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n, \quad x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n.$$