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form kn if \( F_n \mod F_n = 0 \). First we have \( F_n \mod F_n^2 = F_n \); that’s not zero. Next we have

\[
F_{2n} = F_n F_{n+1} + F_{n-1} F_n \equiv 2 F_n F_{n+1} \quad (\text{mod } F_n^2),
\]

by (6.108), since \( F_{n+1} \equiv F_{n-1} \) (mod \( F_n \)). Similarly

\[
F_{2n+1} = F_{n+1}^2 + F_n^2 \equiv F_{n+1}^2 \quad (\text{mod } F_n^2).
\]

This congruence allows us to compute

\[
F_{3n} = F_{2n+1} F_n + F_{2n} F_{n-1} \\
\equiv F_{n+1}^2 F_n + (2 F_n F_{n+1}) F_{n+1} = 3 F_{n+1}^2 F_n \quad (\text{mod } F_n^2);
\]

\[
F_{3n+1} = F_{2n+1} F_{n+1} + F_{2n} F_n \\
\equiv F_{n+1}^2 + (2 F_n F_{n+1}) F_n \equiv 3 F_{n+1} \quad (\text{mod } F_n^2).
\]

In general, we find by induction on \( k \) that

\[
F_{kn} \equiv k F_n F_{n+1}^{k-1} \quad \text{and} \quad F_{kn+1} \equiv F_{n+1}^k \quad \text{(mod } F_n^2)\).
\]

Now \( F_{n+1} \) is relatively prime to \( F_n \), so

\[
F_{kn} \equiv 0 \quad (\text{mod } F_n^2) \iff k F_n \equiv 0 \quad (\text{mod } F_n^2) \iff k \equiv 0 \quad (\text{mod } F_n).
\]

We have proved Matijasevich’s lemma.

One of the most important properties of the Fibonacci numbers is the special way in which they can be used to represent integers. Let’s write

\[
j \gg k \iff j \geq k + 2. \quad (6.112)
\]

Then every positive integer has a unique representation of the form

\[
n = F_{k_1} + F_{k_2} + \cdots + F_{k_r}, \quad k_1 \gg k_2 \gg \cdots \gg k_r \gg 0. \quad (6.113)
\]

(This is “Zeckendorf’s theorem” [201], [312].) For example, the representation of one million turns out to be

\[
1000000 = 832040 + 121393 + 46368 + 144 + 55 \\
= F_{30} + F_{26} + F_{24} + F_{12} + F_{10}.
\]

We can always find such a representation by using a “greedy” approach, choosing \( F_{k_1} \) to be the largest Fibonacci number \( \leq n \), then choosing \( F_{k_2} \) to be the largest that is \( \leq n - F_{k_1} \), and so on. (More precisely, suppose that