(c) Find the first three nonzero terms in each of two linearly independent solutions about $x = 0$.

13. $xy'' + y' - y = 0$

14. $xy'' + 2xy' + 6e^xy = 0$; see Problem 1

15. $x(x - 1)y'' + 6x^2y' + 3y = 0$; see Problem 3

16. $xy'' + y = 0$

17. $x^2y'' + (\sin x)y' - (\cos x)y = 0$

18. Show that 

$$(\ln x)y'' + \frac{1}{2}y' + y = 0$$

has a regular singular point at $x = 1$. Determine the roots of the indicial equation at $x = 1$.

Determine the first three nonzero terms in the series $\sum_{n=0}^{\infty} a_n(x - 1)^{r+n}$ corresponding to the larger root. Take $x - 1 > 0$. What would you expect the radius of convergence of the series to be?

19. In several problems in mathematical physics (for example, the Schrödinger equation for a hydrogen atom) it is necessary to study the differential equation

$$x(1 - x)y'' + [\gamma - (1 + \alpha + \beta)x]y' - \alpha\beta y = 0, \quad (i)$$

where $\alpha$, $\beta$, and $\gamma$ are constants. This equation is known as the hypergeometric equation.

(a) Show that $x = 0$ is a regular singular point, and that the roots of the indicial equation are $0$ and $1 - \gamma$.

(b) Show that $x = 1$ is a regular singular point, and that the roots of the indicial equation are $0$ and $\gamma - \alpha - \beta$.

(c) Assuming that $1 - \gamma$ is not a positive integer, show that in the neighborhood of $x = 0$ one solution of (i) is

$$y_1(x) = 1 + \frac{\alpha\beta}{\gamma - 1!}x + \frac{\alpha(\alpha + 1)\beta(\beta + 1)}{\gamma(\gamma + 1)!}x^2 + \cdots.$$ 

What would you expect the radius of convergence of this series to be?

(d) Assuming that $1 - \gamma$ is not an integer or zero, show that a second solution for $0 < x < 1$ is

$$y_2(x) = x^{1-\gamma} \left[ 1 + \frac{(\alpha - \gamma + 1)(\beta - \gamma + 1)}{(2 - \gamma)!}x + \frac{(\alpha - \gamma + 1)(\alpha - \gamma + 2)(\beta - \gamma + 1)(\beta - \gamma + 2)}{(2 - \gamma)(3 - \gamma)2!}x^2 + \cdots \right].$$ 

(e) Show that the point at infinity is a regular singular point, and that the roots of the indicial equation are $\alpha$ and $\beta$. See Problem 21 of Section 5.4.

20. Consider the differential equation

$$x^3y'' + \alpha xy' + \beta y = 0,$$

where $\alpha$ and $\beta$ are real constants and $\alpha \neq 0$.

(a) Show that $x = 0$ is an irregular singular point.

(b) By attempting to determine a solution of the form $\sum_{n=0}^{\infty} a_nx^{r+n}$, show that the indicial equation for $r$ is linear, and consequently there is only one formal solution of the assumed form.

(c) Show that if $\beta/\alpha = -1, 0, 1, 2, \ldots$, then the formal series solution terminates and therefore is an actual solution. For other values of $\beta/\alpha$ show that the formal series solution has a zero radius of convergence, and so does not represent an actual solution in any interval.