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\[
\sum 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + 13x^6 + 21x^7 + 34x^8 + \cdots
\]

\[= \frac{1 - x - x^2}{1 - x - 2x^2} \]  \hspace{1cm} \text{per Trinomium 1 - x - xx.} \]

-A. de Moivre [64]

“The quantities \( r, s, t \), which show the relation of the terms, are the same as those in the denominator of the fraction. This property, however obvious it may be, M. DeMoivre was the first that applied it to use, in the solution of problems about infinite series, which otherwise would have been very intricate.”

-J. Stirling [281]

refers to \( F_2 \). Otherwise the two least significant digits will be 01, and we change them to \( 10 \) (thereby adding \( F_3 = F_2 = 1 \)). Finally, we must “carry” as much as necessary by changing the digit pattern ‘011’ to ‘100’ until there are no two 1’s in a row. (This carry rule is equivalent to replacing \( F_{m+1} + F_m \) by \( F_{m+2} \).) For example, to go from \( 5 = (1000)_r \) to \( 6 = (1001)_r \) or from \( 6 = (1001)_r \) to \( 7 = (1010)_r \) requires no carrying; but to go from \( 7 = (1010)_r \) to \( 8 = (10000)_r \) we must carry twice.

So far we’ve been discussing lots of properties of the Fibonacci numbers, but we haven’t come up with a closed formula for them. We haven’t found closed forms for Stirling numbers, Eulerian numbers, or Bernoulli numbers either; but we were able to discover the closed form \( \frac{n}{n!} \) for harmonic numbers. Is there a relation between \( F_n \) and other quantities we know? Can we “solve” the recurrence that defines \( F_n \)?

The answer is yes. In fact, there’s a simple way to solve the recurrence by using the idea of generating function that we looked at briefly in Chapter 5. Let’s consider the infinite series

\[
F(z) = F_0 + F_1 z + F_2 z^2 + \cdots = \sum_{n=0}^{\infty} F_n z^n.
\]

If we can find a simple formula for \( F(z) \), chances are reasonably good that we can find a simple formula for its coefficients \( F_n \).

In Chapter 7 we will focus on generating functions in detail, but it will be helpful to have this example under our belts by the time we get there. The power series \( F(z) \) has a nice property if we look at what happens when we multiply it by \( z \) and by \( z^2 \):

\[
z F(z) = F_0 z + F_1 z^2 + F_2 z^3 + F_3 z^4 + F_4 z^5 + \cdots,
\]

\[
z^2 F(z) = F_0 z^2 + F_1 z^3 + F_2 z^4 + F_3 z^5 + \cdots.
\]

If we now subtract the last two equations from the first, the terms that involve \( z^2 \), \( z^3 \), and higher powers of \( z \) will all disappear, because of the Fibonacci recurrence. Furthermore the constant term \( F_0 \) never actually appeared in the first place, because \( F_0 = 0 \). Therefore all that’s left after the subtraction is \( (F_1 - F_0)z \), which is just \( z \). In other words,

\[
F(z) - z F(z) - z^2 F(z) = z,
\]

and solving for \( F(z) \) gives us the compact formula

\[
F(z) = \frac{z}{1 - z - z^2}.
\]

(6.117)