(4–2). Essentially there are two types of $N_F$ orbits in $\mathbb{R}^4$. The first type of $N_F$ orbit corresponds to a "curved conic section", called a regular horocycle. The projection map of $\varphi$ corresponding to a regular horocycle is zero. (This statement is called the First Cusp Vanishing Theorem in [13].) This follows essentially from the fact that $\hat{\varphi}$ is continuous on $\mathbb{R}^4$ and that $\hat{\varphi}$ vanishes on the light cone. The second type of $N_F$ orbit is the affine type discussed above. The contribution of the horocyclic projection maps of $\varphi$ corresponding to the affine type $N_F$ orbit is then determined by Lemma 4.1.

Then using Corollary to Lemma 4.1 and Theorem 4.2, we have

**Corollary to Theorem 4.2.** Let $\varphi \in F_{\delta}(s^2 - 2s)_{\mathbb{R} \times K}$ with $s > \frac{1}{2}k$. Then if $p_F$ is compatible with $L$ and dim $F = b - 1$ or $b$, the constant term of $\Theta_{\delta}^I(G, g)$ in the direction of $N_{\mathbb{R} \times K} \cap \gamma O(Q)_{\mathbb{R} \times K}$ vanishes. In particular if $b = 2$ and $\varphi \in F_{\delta}(s^2 - 2s)_{\mathbb{R} \times K}$, then $\Theta_{\delta}^I(G, g)$ is a cuspidal form on $O(Q)$ relative to $O(Q)_L$.

**Remark 4.2.** Let $\varphi \in F_{\delta}(s^2 - 2s)_{\mathbb{R} \times K}$ with $s > \frac{1}{2}k$. Let $p_F$ be compatible with $L$. If dim $F = b - 1$ (when either $a = 2$ or $a = b$) or dim $F = b$ (when either $b = a - 1$ or $b = a$), then the constant term of $\Theta_{\delta}^I(G, g)$ in the direction of $N_{\mathbb{R} \times K} \cap \gamma O(Q)_{\mathbb{R} \times K}$ vanishes. In particular if $a = 2$, $b = 1$ or 2, and $\varphi \in F_{\delta}(s^2 - 2s)_{\mathbb{R} \times K}$, then $\Theta_{\delta}^I(G, g)$ is a cuspidal form on $O(Q)$ relative to $O(Q)_L$.

The case $b = 2$ is thus the main case of interest. In particular we know that $O(Q)/K_1$, where $K_1 = O(a) \times SO(2)$, is then a Hermitian symmetric space. Let $\mathcal{F} = \mathfrak{f} + \mathfrak{p}$ be the Cartan decomposition of the Lie algebra of $O(Q)$. Then complexifying $\mathcal{F}$ to $\mathcal{F}_C = \mathcal{F} \otimes_{\mathbb{R}} \mathbb{C}$, we have the direct sum $\mathcal{F}_C = \mathfrak{f}_C \oplus p^+ \oplus p^-$, where $p^+$ and $p^-$ span the holomorphic and antiholomorphic tangent vectors at the "origin" in $O(Q)/K_1$. We recall the construction of a family of holomorphic discrete series representations of $O(Q)$. We let $\chi_a : K_1 \to S'$ be the unitary character on $K_1$ which is trivial on $O(a)$ and maps

$$SO(2) = \left\{ \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \right\} : -\pi < \theta \leq \pi$$

to $e^{-\sqrt{-1} \theta}$. Then we form the holomorphically induced representation space

$$\mathcal{H}(O(Q)/K_1, \chi_a) = \left\{ \varphi : O(Q) \to C \mid \varphi \in C^\infty(O(Q)), \varphi(gk) = \varphi(g)\chi_a(k) \right\}$$

for all $g \in O(Q), k \in K_1$, $\varphi \ast W \equiv 0$ for all $W \in p^+$ and

$$\int_{O(Q)/K_1} |\varphi(g)|^2 \, d\tau(g) < \infty$$

with $\ast$ convolution on the right and $d\tau$ some $O(Q)$ invariant measure on $O(Q)/K_1$. Then we have

**Theorem 4.3 ([13], [15]).** The representation of $O(Q)$ in $\mathcal{A}_s$ (see Remark 2.1) is equivalent to the "holomorphic" induced representation of $O(Q)$ in $\mathcal{H}(O(Q)/K_1, \chi_{s_2})$, where $s_2 = |s| + \frac{1}{2}k - 2 (\text{ if } \text{ denotes } C^\infty \text{ vectors}).$

5. **The Shimura-Niwa correspondence.** Having determined the cuspidal properties of $\Theta_{\delta}^I$ in both variables ($\varphi \in F_{\delta}(s^2 - 2s)_{\mathbb{R} \times K}$ with $s > \frac{1}{2}k$), we now go back to
the question of using $\Theta^L_\phi$ as a kernel function to set up a correspondence between modular forms on the groups involved. In particular we let $\phi$ be as in Example 3–1, and consider $\tilde{\Theta}_\phi^L(z, g)$ (see (3–7)). Then we let $f$ be a holomorphic cusp form of weight $s$ ($s > \frac{1}{2}k$) on $H$ satisfying $f(\gamma \cdot z) = \nu_\gamma(\gamma)(c_\gamma z + d_\gamma)^s f(z)$ with $\gamma \in \Delta_{N_L}$. As in §1 we consider the Petersson inner product of $f$ with $\tilde{\Theta}_\phi^L$:

\begin{equation}
\langle \tilde{\Theta}_\phi^L(\cdot, g) | f(\cdot) \rangle = \int_{\Delta_{N_L} \backslash H} \tilde{\Theta}_\phi^L(z, g) f(z) \operatorname{Im} z |^{-2} \ dx \ dy.
\end{equation}

The definition given in (5–1) does not, at first sight, coincide with (1–8), the Petersson inner product $\langle \tilde{\Theta}_\phi^L(\cdot, g) | f(\cdot) \rangle$, where $\tilde{\Theta}_\phi^L$ is the $\theta$ series constructed from the Schwartz function $P(X)e^{-|X|^2}$ (in keeping with the notation of §1, we note that if $\eta \in L$, then we drop the subscript $\eta$ from $\tilde{\Theta}_\phi^L$). In particular we must determine the relationship of $\tilde{\Theta}_\phi^L(\cdot, \cdot)$ to $\tilde{\Theta}_\phi^L(\cdot, \cdot)$ defined above. But we know that the map $\phi \to \tilde{\Theta}_\phi^L(\cdot, \cdot)$ is an $\tilde{S}_L \times O(Q)$ infinitesimal intertwining map from certain $\tilde{S}_L \times O(Q)$ stable subspaces of $L^2(R^k)$ to the space of $C^\infty$ functions on $\tilde{S}_L \times O(Q)$. Thus perhaps the appropriate problem to analyze is the following: for a given $K \times K$ finite function $\phi$ and the projection $\phi^\circ \phi^\circ \phi^\circ$ onto the discrete spectrum of $\tilde{S}_L \times O(Q)$ in $L^2(R^k)$, what is the relation between the $\theta$ series $\tilde{\Theta}_\phi$ and $\tilde{\Theta}_\phi^L$. This we now do partially in

**Lemma 5.1 [16]**. Let $F$ be a $K$ finite Schwartz function in $L^2(R^k)$ and suppose that

\begin{equation}
\pi_m(k(\theta, \varepsilon)) F = e^{-|s\xi|^2} e^{i\xi F} \text{ for all } -\pi < \theta \leq \pi \text{ and } \varepsilon = \pm 1.
\end{equation}

Let $s' > \frac{1}{2}k + 1$. Let $P_{s'}^+$ be the projection of $F$ onto the subspace $F_{s'}(s'^2 - 2s')$. Then

\begin{equation}
\langle \tilde{\Theta}_\phi^L(\cdot, g) | f(\cdot) \rangle = \langle \tilde{\Theta}_\phi^L(\cdot, g) \rangle.
\end{equation}

**Remark 5.1.** If $s < -(\frac{1}{2}k + 1)$, then a similar statement is valid where $P_{s'}^+$ is replaced by $P_{s'}^-$, the projection of $F$ onto $F_{s'}(s'^2 - 2s')$.

Thus the two correspondences (1–8) and (5–1) are essentially the same.

The next main problem is to characterize the image of the map $f \mapsto \langle \tilde{\Theta}_\phi^L(\cdot, g) | f(\cdot) \rangle$ as $f$ varies in the space $[\Delta_{N_L}, s, \nu_0] = \{ f : H \to \mathcal{C} | f \text{ holomorphic, } f(\gamma \cdot z) = \nu_\gamma(\gamma)(c_\gamma z + d_\gamma)^s f(z) \text{ for all } \gamma \in \Delta_{N_L}, z \in H, \text{ and } f \text{ vanishes at the cusp points of } \Delta_{N_L} \text{ on } Q \cup \{ \infty \} \}$, the holomorphic cusp forms of weight $s$ and multiplier $\nu_0$. The first trivial observation is that if we let

\begin{equation}
f(z) = G_s(z) = \sum_{\gamma \in (\Delta_{N_L})/\Delta_{N_L}} \left( \frac{1}{c_\gamma z + d_\gamma} \right)^s \nu_\gamma(\gamma) e^{s - i\pi(\gamma)} z,
\end{equation}

the Eisenstein-Poincaré series, then

\begin{equation}
\langle \tilde{\Theta}_\phi^L(\cdot, g) | G_s(\cdot) \rangle = c_{1} \cdot \varphi_\phi^n(g),
\end{equation}

with $c_1$ a nonzero constant independent of $g$ and $n$. However we know since $s > \frac{1}{2}k$, we know that the functions $G_s$ span $[\Delta_{N_L}, s, \nu_0]$, and hence the space $\langle \tilde{\Theta}_\phi^L(\cdot, g) | f(\cdot) \rangle | f \in [\Delta_{N_L}, s, \nu_0] \rangle$ is exactly the complex linear span of the $\varphi_\phi^n$ as $n \geq 1$.

From Corollary to Theorem 4.2 we know the cases when all $\varphi_\phi^n (n \geq 1)$ are cusp forms on $O(Q)$ relative to $O(Q)_L$. And the most general Fourier coefficient of $\varphi_\phi^n$ is difficult to describe arithmetically, since $\varphi_\phi^n$ is essentially very much like a Poincaré form. However using the results of §4, it is possible to determine for which $f \in [\Delta_{N_L}, s, \nu_0]$, $\langle \tilde{\Theta}_\phi^L(\cdot, g) | f(\cdot) \rangle$ will be a cusp form on $O(Q)$ relative to $O(Q)_L$. 