21. Consider the differential equation

\[ y'' + \frac{\alpha}{x^s} y' + \frac{\beta}{x^t} y = 0, \tag{i} \]

where \( \alpha \neq 0 \) and \( \beta \neq 0 \) are real numbers, and \( s \) and \( t \) are positive integers that for the moment are arbitrary.

(a) Show that if \( s > 1 \) or \( t > 2 \), then the point \( x = 0 \) is an irregular singular point.

(b) Try to find a solution of Eq. (i) of the form

\[ y = \sum_{n=0}^{\infty} a_n x^{r+n}, \quad x > 0. \tag{ii} \]

Show that if \( s = 2 \) and \( t = 2 \), then there is only one possible value of \( r \) for which there is a formal solution of Eq. (i) of the form (ii).

(c) Show that if \( s = 1 \) and \( t = 3 \), then there are no solutions of Eq. (i) of the form (ii).

(d) Show that the maximum values of \( s \) and \( t \) for which the indicial equation is quadratic in \( r \) [and hence we can hope to find two solutions of the form (ii)] are \( s = 1 \) and \( t = 2 \). These are precisely the conditions that distinguish a “weak singularity,” or a regular singular point, from an irregular singular point, as we defined them in Section 5.4.

As a note of caution we should point out that while it is sometimes possible to obtain a formal series solution of the form (ii) at an irregular singular point, the series may not have a positive radius of convergence. See Problem 20 for an example.

### 5.8 Bessel’s Equation

In this section we consider three special cases of Bessel’s equation,

\[ x^2 y'' + xy' + (x^2 - \nu^2) y = 0, \tag{1} \]

where \( \nu \) is a constant, which illustrate the theory discussed in Section 5.7. It is easy to show that \( x = 0 \) is a regular singular point. For simplicity we consider only the case \( x > 0 \).

**Bessel Equation of Order Zero.** This example illustrates the situation in which the roots of the indicial equation are equal. Setting \( \nu = 0 \) in Eq. (1) gives

\[ L[y] = x^2 y'' + xy' + x^2 y = 0. \tag{2} \]

Substituting

\[ y = \phi(r, x) = a_0 x^r + \sum_{n=1}^{\infty} a_n x^{r+n}, \tag{3} \]

\(^{12}\)Friedrich Wilhelm Bessel (1784 –1846) embarked on a career in business as a youth, but soon became interested in astronomy and mathematics. He was appointed director of the observatory at Königsberg in 1810 and held this position until his death. His study of planetary perturbations led him in 1824 to make the first systematic analysis of the solutions, known as Bessel functions, of Eq. (1). He is also famous for making the first accurate determination (1838) of the distance from the earth to a star.