We can find $\alpha$ and $\beta$ in several ways, one of which uses a slick trick: Let us introduce a new variable $w$ and try to find the factorization

$$w^2 - wz - z^2 = (w - \alpha z)(w - \beta z).$$

Then we can simply set $w = 1$ and we'll have the factors of $1 - z - z^2$. The roots of $w^2 - wz - z^2 = 0$ can be found by the quadratic formula; they are

$$z = \frac{-1 \pm \sqrt{1 + 4z^2}}{2} = \frac{1 \pm \sqrt{5}}{2}z.$$

Therefore

$$w^2 - wz - z^2 = \left(w - \frac{1 + \sqrt{5}}{2}z\right)\left(w - \frac{1 - \sqrt{5}}{2}z\right).$$

and we have the constants $\alpha$ and $\beta$ we were looking for.

The number $(1 + \sqrt{5})/2 \approx 1.61803$ is important in many parts of mathematics as well as in the art world, where it has been considered since ancient times to be the most pleasing ratio for many kinds of design. Therefore it has a special name, the golden ratio. We denote it by the Greek letter $\phi$, in honor of Phidias who is said to have used it consciously in his sculpture. The other root $(1 - \sqrt{5})/2 = -1/\phi \approx -0.61803$ shares many properties of $\phi$, so it has the special name $\hat{\phi}$, "phi hat!" These numbers are roots of the equation $w^2 - w - 1 = 0$, so we have

$$\phi^2 = \phi + 1; \quad \hat{\phi}^2 = -\hat{\phi} + 1. \quad (6.121)$$

(More about $\phi$ and $\hat{\phi}$ later.)

We have found the constants $\alpha = \phi$ and $\beta = \hat{\phi}$ needed in (6.119); now we merely need to find $A$ and $B$ in (6.120). Setting $z = 0$ in that equation tells us that $B = -A$, so (6.120) boils down to

$$\phi A + \hat{\phi} A = 1.$$

The solution is $A = 1 / (\phi - \hat{\phi}) = 1 / \sqrt{5}$; the partial fraction expansion of (6.117) is therefore

$$F(z) = \frac{1}{\sqrt{5}} \left( \frac{1}{1 - \phi z} - \frac{1}{1 - \hat{\phi} z} \right). \quad (6.122)$$

Good, we've got $F(z)$ right where we want it. Expanding the fractions into power series as in (6.118) gives a closed form for the coefficient of $z^n$:

$$F_n = \frac{1}{\sqrt{5}} (\phi^n - \hat{\phi}^n). \quad (6.123)$$

(This formula was first published by Leonhard Euler [91] in 1765, but people forgot about it until it was rediscovered by Jacques Binet [25] in 1843.)