Before we stop to marvel at our derivation, we should check its accuracy. For \( n = 0 \) the formula correctly gives \( F_0 = 0 \); for \( n = 1 \), it gives \( F_1 = (\phi - \overline{\phi})/\sqrt{5} \), which is indeed 1. For higher powers, equations (6.121) show that the numbers defined by (6.123) satisfy the Fibonacci recurrence, so they must be the Fibonacci numbers by induction. (We could also expand \( \phi^n \) and \( \overline{\phi}^n \) by the binomial theorem and chase down the various powers of \( \sqrt{5} \); but that gets pretty messy. The point of a closed form is not necessarily to provide us with a fast method of calculation, but rather to tell us how \( F_n \) relates to other quantities in mathematics.)

With a little clairvoyance we could simply have guessed formula (6.123) and proved it by induction. But the method of generating functions is a powerful way to discover it; in Chapter 7 we’ll see that the same method leads us to the solution of recurrences that are considerably more difficult. Incidentally, we never worried about whether the infinite sums in our derivation of (6.123) were convergent; it turns out that most operations on the coefficients of power series can be justified rigorously whether or not the sums actually converge \([151]\). Still, skeptical readers who suspect fallacious reasoning with infinite sums can take comfort in the fact that equation (6.123), once found by using infinite series, can be verified by a solid induction proof.

One of the interesting consequences of (6.123) is that the integer \( F_n \) is extremely close to the irrational number \( \phi^n/\sqrt{5} \) when \( n \) is large. (Since \( \overline{\phi} \) is less than 1 in absolute value, \( \overline{\phi}^n \) becomes exponentially small and its effect is almost negligible.) For example, \( F_{10} = 55 \) and \( F_{11} = 89 \) are very near

\[
\frac{\phi^{10}}{\sqrt{5}} \approx 55.00364 \quad \text{and} \quad \frac{\phi^{11}}{\sqrt{5}} \approx 88.99775.
\]

We can use this observation to derive another closed form,

\[
F_n = \left\lfloor \frac{\phi^n + 1}{2} \right\rfloor = \frac{\phi^n}{\sqrt{5}} \text{ rounded to the nearest integer,} \quad (6.124)
\]

because \( \phi^n/\sqrt{5} < \frac{1}{2} \) for all \( n \geq 0 \). When \( n \) is even, \( F_n \) is a little bit less than \( \phi^n/\sqrt{5} \); otherwise it is a little greater.

Cassini’s identity (6.103) can be rewritten

\[
\frac{F_{n+1}}{F_n} - \frac{F_n}{F_{n-1}} = \frac{(-1)^n}{F_{n-1}F_n}
\]

When \( n \) is large, \( 1/F_{n-1}F_n \) is very small, so \( F_{n+1}/F_n \) must be very nearly the same as \( F_n/F_{n-1} \); and (6.124) tells us that this ratio approaches \( \phi \). In fact, we have

\[
F_{n+1} = \phi F_n + \overline{\phi}^n. \quad (6.125)
\]