The second solution of the Bessel equation of order zero is found by setting $a_0 = 1$ and substituting for $y_1(x)$ and $a^2_{2m}(0) = b_{2m}(0)$ in Eq. (23) of Section 5.7. We obtain

$$y_2(x) = J_0(x) \ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{2^{2m} (m!)^2} x^{2m}, \quad x > 0. \quad (10)$$

Instead of $y_2$, the second solution is usually taken to be a certain linear combination of $J_0$ and $y_2$. It is known as the Bessel function of the second kind of order zero and is denoted by $Y_0$. Following Copson (Chapter 12), we define

$$Y_0(x) = \frac{2}{\pi} \left[ y_2(x) + (\gamma - \ln 2) J_0(x) \right]. \quad (11)$$

Here $\gamma$ is a constant, known as the Euler–Mácheroni (1750–1800) constant; it is defined by the equation

$$\gamma = \lim_{n \to \infty} (H_n - \ln n) \cong 0.5772. \quad (12)$$

Substituting for $y_2(x)$ in Eq. (11), we obtain

$$Y_0(x) = \frac{2}{\pi} \left[ \left( \gamma + \ln \frac{x}{2} \right) J_0(x) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{2^{2m} (m!)^2} x^{2m} \right], \quad x > 0. \quad (13)$$

The general solution of the Bessel equation of order zero for $x > 0$ is

$$y = c_1 J_0(x) + c_2 Y_0(x).$$

Note that $J_0(x) \to 1$ as $x \to 0$ and that $Y_0(x)$ has a logarithmic singularity at $x = 0$; that is, $Y_0(x)$ behaves as $(2/\pi) \ln x$ when $x \to 0$ through positive values. Thus if we are interested in solutions of Bessel’s equation of order zero that are finite at the origin, which is often the case, we must discard $Y_0$. The graphs of the functions $J_0$ and $Y_0$ are shown in Figure 5.8.2.

It is interesting to note from Figure 5.8.2 that for $x$ large both $J_0(x)$ and $Y_0(x)$ are oscillatory. Such a behavior might be anticipated from the original equation; indeed it

![FIGURE 5.8.2 The Bessel functions of order zero.](image)

[1] Other authors use other definitions for $Y_0$. The present choice for $Y_0$ is also known as the Weber (1842–1913) function.